We propose new techniques for understanding agents’ valuations. Our classification into “demand types” incorporates existing definitions (such as substitutes, complements, “strong substitutes”, etc.), and permits additional distinctions. It yields easy-to-check necessary and sufficient conditions for the existence of a competitive equilibrium for indivisible goods. Our conditions generalise many existing results. Contrary to popular belief, equilibrium for indivisible goods does not depend on substitutes preferences; indeed, there are more classes of purely-complements preferences than classes of purely-substitutes preferences for which competitive equilibrium always exists. We also study cases in which equilibrium is not guaranteed for an entire “demand type”. Whether or not equilibrium exists for every possible market supply, for a specific set of individual valuations, can often be checked by simply counting the number of intersection points of the geometric objects we study! Finally, our methods open the way to new results about stable matching in many-player matching models, and further develop the Product-Mix Auction, introduced by the Bank of England in response to the financial crisis.

Keywords: consumer theory; equilibrium existence; general equilibrium; competitive equilibrium; duality; indivisible goods; geometry; tropical geometry; convex geometry; auction; product mix auction; product-mix auction; substitute; complement; demand type; matching

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*London School of Economics, UK; e.c.baldwin@lse.ac.uk
†Nuffield College, Oxford University, UK; paul.klemperer@economics.ox.ac.uk
‡This paper extends and supersedes much of the material in Baldwin and Klemperer (2014). However, large parts of Sections 2.5, 6.2, and 6.3.1 of that paper will be incorporated and developed in Baldwin and Klemperer (in preparation-a), and Sections 4.2, 4.3, and 5, form the main part Baldwin and Klemperer (in preparation-b). This work was supported by ESRC grant ES/L003058/1. Acknowledgements to be completed later.
1 Introduction

This paper introduces a new way of thinking about demand for indivisible goods, and obtains new results about the existence of competitive equilibrium.¹

“Demand types”

Our first key idea is to classify economic agents’ individual and aggregate valuations into “demand types”. A “demand type” is defined by a list of vectors that give the possible ways in which the individual or aggregate demand can change in response to a small generic price change. These vectors are analogous to the rows of a Slutsky matrix (the list of vectors, each of which is the set of derivatives of demand with respect to the price of one of a set of divisible goods); with indivisibilities the dimensionality is low enough that we can characterize a class of valuations globally in this way.²

So the vectors that define a “demand type” specify the possible comparative statics of any demand of that “type”, and thus much of what economists think important about valuations. For example, a purchaser of lenses and spectacle frames who is interested in having spare pairs might always buy in the ratio 2:1, so always increases or reduces her demand for lenses and frames in this ratio, whatever the individual prices of the goods: we will describe any preferences of this “type” by the “demand type” ±{(2,1)}.

As another example, a consumer who views apples and bananas as substitutes would have preferences of “demand type” ±{(1,0),(0,1),(-1,1)} if, when prices change slightly, her bundle only ever changes in the direction of adding or subtracting an apple, or of adding or subtracting a banana, or of substituting one piece of one kind of fruit for one piece of the other kind.³

Our classification is parsimonious. For example, the “type” that comprises all possible substitutes preferences for indivisible goods is the set of all vectors with at most one positive integer entry, at most one negative integer entry, and all other entries zero; the “type” that is all complements preferences for indivisible goods is the set of all vectors in which all the non-zero entries (of which there may be any number) are integers of the same sign; the class of all “strong substitutes”⁴ preferences for n goods is a “demand type” with just n(n + 1) vectors.

Our classification clarifies the relationships between different classes of preferences. For example, the “demand types” descriptions above show clearly why the conditions for three or more indivisible goods to all be (ordinary) substitutes are far more restrictive than the conditions for them to all be complements—although they are, of course, symmetric in the two-good case.

Our classification is also easy to work with, and yields powerful new results:

Equilibrium existence

¹Our approach applies to agents who buy and/or sell, and also to some matching models. Baldwin and Klemperer (in preparation-c) shows our techniques also help analyse divisible goods.

²We assume, as is standard in the indivisible-goods literature, that preferences are quasilinear, so there is no distinction between compensated and uncompensated demand.

³For example, in an auction in which goods’ characteristics suggest natural rates of substitution, bidders can be asked to express preferences that come from the corresponding “demand type”. Indeed the original version of the Bank of England’s Product-Mix Auction had one-for-one substitution built into its design (see Klemperer, 2008, 2010).

⁴Milgrom and Strulovici’s (2009) “strong substitutes” generalised many existing valuation structures.
We give a simple necessary and sufficient condition, which generalises existing results, about whether or not competitive equilibrium always exists, whatever is the market supply, if all agents’ valuations are drawn from a class of valuations (i.e. are of a given “demand type”).

Our condition can easily be checked using the determinants of sets of the vectors in the “demand type”. So we can quickly see whether any demand structure guarantees equilibrium existence. Several well-known results are easy special cases. Moreover, our results demolish the popular perception that the existence of equilibrium with indivisible goods depends on substitutes relationships, or basis changes of substitutes relationships.\(^5\) Indeed every demand type for which equilibrium is guaranteed can be obtained as a basis change of a demand type involving only complementary relationships (and for which equilibrium is also guaranteed)—and the corresponding result is not true for substitute preferences.

Our geometric methods also give beautiful answers to whether or not competitive equilibrium exists for any market supply, for a specific set of agents’ valuations, when they are not all drawn from a “demand type” for which existence is always guaranteed.

**Description of the paper**

The reason our “demand types” are a mathematically convenient way to categorize valuations is because the vectors they comprise describe how price space is divided into the different regions in which an agent demands different bundles: this division creates precisely the geometric structures studied in “tropical” geometry.\(^6\) So we can apply the tools of convex and tropical geometry. The duality between the geometric object representing a valuation in price space, and the geometric object corresponding to the same valuation in quantity space, is particularly fruitful.

So we begin, in Section 2, by translating some existing mathematics literature into economics. We describe the properties of a “tropical hypersurface”, a geometric object which contains precisely those points at which the agent is indifferent between two or more bundles. Moreover, we observe that any geometric structure of this kind corresponds to a valuation function, so we can develop our understanding of demand by working directly with these geometric objects; we believe this is the first paper to do this. We then explore duality for indivisible demand.

Section 3 defines a “demand type” by the set of vectors describing the ways in which the bundles demanded by the agent change with prices. It is then elementary to check whether a demand type is, for example, substitutes, or complements, or “strong substitutes”, or “gross substitutes and complements”, etc.\(^7\)

Importantly, the “demand type” of the aggregate valuation of multiple agents is simply the union of the vectors of the individual “demand types”. This observation

\(^5\)That is, there may not exist any repackaging of goods after which the agents’ valuations are for substitutes. (See Section 4.3.2.)

\(^6\)Tropical geometry is a non-Euclidean branch of algebraic geometry recently developed by, among others, Mikhalkin (2004, 2005). We believe it has not previously been applied to economics. Goeree and Kushnir (2012) use convex geometry (see, e.g., Rockafellar, 1970), on which tropical geometry builds, in a different context. However, Danilov and Koshevoy and their co-authors’ methods of discrete convex analysis have closer connections to ours (see, in particular, Danilov et al., 2001, Danilov et al., 2003 and Danilov and Koshevoy, 2004), as we discuss later in the Introduction, and in detail in Section 4.1.

\(^7\)There are software solutions to calculate the relevant vectors, and hence the demand ‘type’ of any valuation.
opens the way to new results about the existence of equilibrium.

Section 4.1 therefore studies competitive equilibrium for “demand types”: whether or not equilibrium exists depends on the nature of the intersections of agents’ tropical hypersurfaces. So the theory of tropical intersection multiplicities inspires our proof that equilibrium always exists for any set of agents who all have concave valuations over $n$ goods (of each of which there may be multiple units) of a given “demand type” if and only if every subset of $n$ of the “type’s” vectors has determinant 0 or ±1 (plus an additional condition if the demand type’s set of vectors is in fewer than $n$ dimensions).

So our necessary and sufficient condition for equilibrium is easy to check, and can be checked for each agent separately. By contrast, Bikhchandani and Mamer’s (1997) and Ma’s (1998) conditions for existence of equilibrium for a set of agents need to be checked against every possible combination of agents—this seems both less practical and to give less insight into why agents’ valuations do or don’t permit equilibrium.

Our sufficient condition yields a class of results, each stating that equilibrium always exists when every individual valuation has a certain property. An example of such a result is that equilibrium always exists when every agent’s valuation is “strong substitutes”. This specific result is not new (see Milgrom and Strulovici, 2009), but it follows immediately from our theorem, as do others, such as some in Kelso and Crawford (1982), Sun and Yang (2006), Hatfield et al. (2013), and Teytelboym (2014), and extensions of many of these.

New properties that guarantee equilibrium are also easy to generate. For example, we exhibit a family of valuations over four goods (and multiple units of each good) involving only complementarities, and for which equilibrium always exists.⁸

Because our condition is also necessary, we can quickly check whether equilibrium will always exist if agents’ valuations are of any particular demand type. It follows easily, for example, that with (multiple units of) three or fewer goods, equilibrium always exists if and only if goods are either “strong substitutes” or a basis change of strong substitutes.⁹ (However, this is not true with four or more distinct goods.) Furthermore, our geometric approach immediately provides an example of failure of equilibrium whenever our condition fails.

This theorem is closely related to the work in a remarkable series of papers by Danilov and Koshevoy and their co-authors. In particular, Danilov et al. (2001) provide a sufficient condition for equilibrium, which is mathematically dual to our sufficient condition. However, our concept of ‘demand types’ shows how this condition can be applied—for example, none of the papers listed above, whose equilibrium existence results are obvious corollaries of our theorem, present their results as applications of Danilov et al., since the latter’s relevance was not clear. Our ‘demand types’ also illuminate the condition’s economic meaning for individual agents and, moreover, they make clear the sense in which the same condition is also necessary for equilibrium, which is not proved in Danilov et al. (2001).¹⁰

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⁸Moreover, it is not a basis change of any substitute preferences.

⁹Observe that our necessity result contrasts with “necessity” results of the kind given in several of the works listed above, which show only that equilibrium always exists if all agents’ valuation functions have a certain property, but may fail if just one valuation function does not.

¹⁰We discuss the relationships to, and other distinctions from, Danilov and Koshevoy and their co-authors’ work in Section 4.1. Analysing and interpreting our concept of “demand types” in price space allows us to develop economic implications further than they do, since they almost exclusively study...
Finally, Section 5 shows that our methods, unlike Danilov et al.’s, also yields additional existence results of a quite different kind, namely about when combinations of specific valuations always yield equilibria (as distinct from when any set of valuations from some class always yield equilibria):

Tropical hypersurfaces can be understood as transformations of “ordinary” geometric objects, and versions of “ordinary” intersection theory apply tropically. In particular, a version of Bézout’s classic theorem that the number of intersections of two polynomials in two dimensions is the product of their degrees unless there are “multiplicities” such as tangencies (and its generalisations, including to higher dimensions) remains true.¹¹ Since, we will see, failures of competitive equilibrium correspond to multiplicities, we can therefore often determine the existence or failure of equilibrium by simply counting the intersections of the tropical hypersurfaces! Furthermore, even when this count does not suffice, our methods yield a recipe for determining whether or not equilibrium always exists for a given set of individual agents’ valuations (for any possible market supply). Moreover, our recipe requires checking the properties of the valuations at only a finite collection of prices, whose number we bound.

Section 6 presents applications: Sections 6.1-6.2 observe that our model encompasses classic models such as Kelso and Crawford (1982), Hatfield et al (2013), and cycles of complements. This clarifies the relationships between these papers, reveals how they can extended, and shows that their equilibrium-existence results are immediate special cases of ours. Section 6.3 uses the fact that equilibrium properties are unaffected by basis changes to show yet more results are easy corollaries of our work. Section 6.4, by contrast, exhibits a new, purely-complements, demand type for which equilibrium is guaranteed, but which is not a simple basis change of any standard demand type.

Section 6.5-6.6 notes that our approach yields new results about when stable matches exist in multi-agent matching models, and has additional applications to the theory of an individual agent’s demand.

Finally, Section 6.7 observes that our geometric techniques can develop extensions to the Bank of England’s “Product-Mix Auction”.¹² Our methods show what kinds of bids are needed to represent different kinds of preferences, analyse the implications of different restrictions on bids, reveal how to efficiently solve for equilibrium (and when it exists), etc.

Section 7 concludes. Appendix A contains proofs of results in the text.

quantity space. Moreover, though our techniques are novel, they are more straightforward than theirs. However, their work deserves far more attention than it has thus far received.

¹¹For example, in “ordinary” geometry two lines intersect once (possibly at infinity). A quadratic and a line intersect twice including intersections in the complex plane (and counting intersections at infinity, and double-counting tangencies). Two quadratics intersect four times (correctly counted), etc.

¹²Bidders in these auctions make sets of either/or bids for alternative objects. The Bank of England represents these bids geometrically as sets of points in multi-dimensional price space.

The then-Governor of the Bank (Mervyn King) told The Economist that the Product-Mix Auction “is a marvellous application of theoretical economics to a practical problem of vital importance to financial markets”; an Executive Director of the Bank described it as “a world first in central banking”, and “potentially a major step forward in practical policies to support financial stability”; and current-Governor Mark Carney announced plans for greater use of the auction, and introduced an updated version endogenising total quantity and permitting more dimensions (i.e., more goods)–see Bank of England, 2010, 2011, Milnes, 2010, Fisher, 2011, Fisher et al., 2011, and The Economist, 2012.

(In principle, of course, the loans that the Central Bank auctions are almost continuously divisible, but we can use some of our same indivisible-good techniques to analyse this auction.)
2 Representing Indivisible Demand Geometrically

2.1 Assumptions, and Tropical Hypersurfaces (THs)

There are \( n \) goods, which come in indivisible units. Each agent has a quasilinear valuation function \( u : A \to \mathbb{R} \) over a finite domain \( A \subseteq \mathbb{Z}^n \) of possible bundles.\(^{13}\) We make no further restrictions on the domain \( A \); it need not be discrete-convex,\(^{14}\) and it may include negative bundles so agents can sell as well as buy. So at a price vector \( p \in \mathbb{R}^n \),\(^{15}\) the agent demands

\[
D_u(p) := \arg \max_{x \in A} \{u(x) - p \cdot x\}.
\]

We will be particularly interested in the prices at which demand changes, that is, those prices at which the agent is indifferent among more than one bundle, namely the set of \( p \) for which \( \#D_u(p) > 1 \). This set (with some additional structure—see Definition 2.1) is known as a ‘tropical hypersurface’ (TH).\(^{16}\) We will see that we have an essentially perfect correspondence between THs and concave valuation functions (Theorem 2.7), but believe ours is the first paper to use THs in economics.

2.2 The Structure of Tropical Hypersurfaces

Fig. 1 shows a simple example of a TH. The agent’s valuations are \( u(0,0) = 0 \), \( u(1,0) = 5 \) and \( u(0,1) = 4 \). So it demands a unique bundle in each of the three unique demand regions (UDRs), but switches between bundles along the three line segments, which together form the TH. In general, a TH is made up of \( (n - 1) \)-dimensional linear components, which we call facets, and which separate the \( n \)-dimensional UDRs from each other.

\(^{13}\) We initially restrict to a single agent. We will later consider a finite set of agents with valuations \( u_j \) on domains \( A_j \), \( j = 1, \ldots, m \), and will then write \( A \) for the “aggregate” domain, \( \{\sum_j x^j \mid x^j \in A_j\} \).

\(^{14}\) \( A \) is discrete-convex if all the integer points in its convex hull, \( \text{Conv} A \), are in \( A \), that is, \( (\text{Conv} A) \cap \mathbb{Z}^n = A \).

\(^{15}\) So different units of the same good always have the same price; we allow negative prices. (We can, of course, model different units of a homogeneous good which are priced independently by treating them as different goods.)

\(^{16}\) See Mikhalkin (2004) and others. In fact, our TH is ‘upside down’ compared with Mikhalkin who considers the non-smooth locus of \( p \mapsto \max_{x \in A} \{x \cdot p - u(x)\} \), but his convention is not universal, and our definition seems the most natural for economics.
A facet is defined to be closed, i.e., to contain its boundary; that boundary is itself made up of (finitely many) \((n-2)\)-dimensional linear components, and this pattern continues on down the dimensions. Any geometric object satisfying this description is called a polyhedral complex. It is also rational if, as will always be the case for us, each of its components can be defined by (linear) equations with integer coefficients. The \(k\)-dimensional components are called \(k\)-cells. So, for example, the TH of Fig. 1 contains three 1-cells, and one 0-cell (where the 1-cells meet). Observe that each 1-cell is the complete set of prices where two specific bundles are demanded, while the 0-cell is the unique price where all three possible bundles are demanded. More generally, any \(k\)-cell is the set of prices at which a particular set of bundles is demanded. Appendix A.1.1 gives a full, formal, taxonomy of THs and their economic interpretation.

Demand is constant in each UDR, since demand cannot switch from one unique bundle to another without passing across a facet. Furthermore, at any price in any facet, the agent is indifferent between the bundles \(x\) and \(x'\) demanded in the UDRs on either side of the facet. That is, \(u(x) - p \cdot x = u(x') - p \cdot x'\) for every \(p\) in any facet, \(F\). So \(p \cdot (x' - x)\) is a constant for \(p \in F\), and the vector \(x' - x\) is therefore normal to \(F\). We call the greatest common divisor of the entries of \(x' - x\) the weight of the facet, \(w(F)\). So \(\frac{1}{w(F)} (x' - x)\) is a primitive integer vector (that is, the greatest common divisor of its entries is 1), and it points from the UDR where \(x'\) is demanded to the UDR where \(x\) is. But since \(F\) is \((n - 1)\) dimensional, its normal direction is unique, so there is a unique primitive integer normal vector pointing from the UDR of \(x'\) to that of \(x\). So if we know \(F\), \(w(F)\) and \(x\), we can derive \(x'\). More generally, therefore, if we know the demand in any one UDR, we can work out the demand in any UDR, if we also know (1) all the facets (i.e., all the prices at which demand is non-unique), and (2) all the facet weights—and (1) and (2) are precisely the information that defines a TH:

**Definition 2.1** (Mikhalkin, 2004, Example 2). The tropical hypersurface \(T_u\) associated with any valuation \(u\) is the weighted rational polyhedral complex such that:

1. its underlying set is \(\{p \in \mathbb{R}^n | \#D_u(p) > 1\}\);
2. the weight \(w_u(F)\) of the facet \(F\) is the integer defined by \(w_u(F)\cdot v_F = x' - x\), in which \(x'\) is demanded in the UDR on one side of \(F\); \(x\) is demanded in the UDR on the other side; and \(v_F\) is the primitive integer normal vector pointing from the former to the latter.

This definition is mathematically equivalent to Mikhalkin’s, but the mathematical literature has not, of course, interpreted them in an economic context (that is, understood the \(D_u(p)\) as demand sets).

### 2.3 The correspondence between specific valuations and THs

If we follow an agent’s demand along a price path that crosses facets but ends where it started, it demands at the end exactly what it demanded at the beginning. So the weights on the facets must satisfy the balancing condition:

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17Note that facets may not ‘cross’; if they would seem to, the part on either side is identified as a distinct facet.
18So in three dimensions, for example, the facets are pieces of plane, whose boundaries are the line segments where they meet; the boundaries of the line segments are the points where line segments meet.
**Definition 2.2** (Mikhalkin, 2004, Definition 3). An \((n-1)\)-dimensional weighted rational polyhedral complex \(\Pi \subseteq \mathbb{R}^n\) is **balanced** if for every every \((n-2)\)-dimensional cell \(G \subseteq \Pi\), the weights \(w(F_j)\) on the facets \(F_1, \ldots, F_l\) that are adjacent to \(G\), and primitive integer normal vectors \(\mathbf{v}_{F_j}\) for these facets that are defined by a rotational direction about \(G\), satisfy

\[
\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0.19
\]

This balancing condition is in fact the only condition that a weighted rational polyhedral complex has to satisfy to be the TH of some valuation function.\(^{20}\)

**Theorem 2.3** (Mikhalkin, 2004, Proposition 2.4; also Mikhalkin, 2005, Theorem 3.15). Suppose that \(\Pi \subseteq \mathbb{R}^n\) is an \((n-1)\)-dimensional balanced weighted rational polyhedral complex.\(^{21}\) Then there exists a finite set \(A \subseteq \mathbb{Z}^n\) and a function \(u : A \rightarrow \mathbb{R}\) such that \(\Pi\) is the TH, \(T_u\).

It follows that a set in \(\mathbb{R}^n\) is the TH of some quasilinear valuation if and only if it is a rational polyhedral complex and there exist weights for the facets such that it is balanced. It is often much easier to develop our ideas and intuitions by working with these geometric objects than by thinking of examples of valuations, and in the subsequent sections we will see how describing the geometry of the objects gives us insights into their economics.

We will be particularly interested in concavity of valuation functions in the standard discrete sense:

**Definition 2.4.** A function \(u : A \rightarrow \mathbb{R}\) is **concave** if \(A\) is a discrete-convex set and \(u\) can be extended to a weakly-concave function on \(\mathbb{R}^n\).

The significance of concavity is that it is a standard result that concave valuations are precisely those for which every possible bundle is demanded at some price, and for which the demand set at any price is discrete-convex (just as for divisible, weakly-concave, valuations, and for essentially the same reasons\(^{22}\)):

**Lemma 2.5.** \(u : A \rightarrow \mathbb{R}\) is concave

iff \(A\) is a discrete-convex set and for all \(x \in A\) there exists \(p\) such that \(x \in D_u(p)\)

iff \(D_u(p)\) is discrete-convex for all \(p\).

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19 This is just the \(n\)-dimensional generalisation of the requirement in 2 dimensions that, when moving in a sufficiently small circle around any point, the weights on any facets crossed be coherent. To choose a rotational direction around \(G\), pick a 2-dimensional affine subspace \(H\) of \(\mathbb{R}^n\) orthogonal to \(G\), such that the intersection of each \(F_j\) with \(H\) is 1-dimensional. The intersection of \(H\) with the TH is then a collection of 1-cells meeting at the 0-cell which is \(G \cap H\). An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around \(G\) in \(\mathbb{R}^n\).

20 There do not necessarily exist weights to balance a general rational polyhedral complex. For example, in two dimensions, consider three points (0-cells), each with three adjacent facets. If each pair of points has an adjacent facet in common, the six weights must satisfy six balancing conditions (three in each of the two dimensions). But since the conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent—which is non-generic.

21 Strictly speaking, of course, \(\Pi\) is a subset of the space \(\mathbb{R}^n\) and has weights. As before, we follow Mikhalkin and the mathematical literature in our presentation.

22 See, e.g., Mas-Colell et al. (1995) pp. 135-8, especially Prop. 5.C.1(v), since a quasilinear valuation is equivalent to a standard profit function with a single-output technology. These results are also clear from considering the example in the next subsection (2.4).
Furthermore, it is easy to see that any non-concave valuation has the same TH as the minimal concave function that weakly exceeds it, since increasing any never-demanded bundle’s value has no effect until the bundle is demanded, and when it is just marginally demanded (when the value function becomes locally affine) all previously-demanded bundles are still demanded at the same prices as previously. At this point the marginally-demanded bundle is demanded at the price(s) at which the agent is indifferent between some other bundles, that is, at all the prices in an existing cell of the TH, which leaves the TH unaltered:

**Lemma 2.6.** Let \( u' \) be the minimal concave function that weakly exceeds \( u \).\(^{23}\) Then \( \mathcal{T}_{u'} = \mathcal{T}_u \).

Clearly, adding a constant to \( u(x) \) leaves the TH unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation.\(^{24}\) So we have full equivalence between THs and concave valuation functions, up to shifts by a constant:

**Theorem 2.7** (Mikhalkin, 2004, Remark 2.3). There is a 1-1 correspondence between THs with an identified ‘demand 0’ UDR, and pairs \((u, A)\), where \( u \) is a concave function on a discrete-convex \( A \) for which \( u(0) = 0 \) and demand is 0 at prices in the ‘demand 0’ UDR.

Importantly, therefore, *any* balanced weighted rational polyhedral complex also corresponds to some concave valuation, so we can develop our understanding of valuations by working directly with these geometric objects. However, we will *not* restrict attention to concave valuations.

### 2.4 Duality; and Subdivided Newton Polytopes (SNPs)

We constructed the TH in price space. We now construct a dual geometric object—the Subdivided Newton Polytope (SNP)—in quantity space.\(^{25}\) This presents much of the same information in a complementary way.\(^{26}\)

Just as in the standard duality construction for a divisible, strictly-concave, valuation, any price vector defines a tangent hyperplane meeting the graph of the agent’s valuation at the agent’s demand set for that price. But in our case, because the demand set sometimes contains more than one bundle, some tangent hyperplanes meet the graph at more than one point.

For example, Fig. 2a shows a valuation function, \( u \), and Fig. 2b gives its graph, using bars to associate a bundle, \( x \), with its valuation, \( u(x) \). We will always present the feasible bundles increasing to the left, and down. This will show the duality between the SNP and the TH most clearly.

The bundles demanded at any given price are those which maximise the valuation with respect to that price, and so are “furthest out” from the origin in the direction

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\(^{23}\)If \( u \)'s domain is not discrete-convex, then \( u' \)'s domain must be the minimum discrete-convex set containing it.

\(^{24}\)Of course, the bundle demanded in each UDR is then increased by the fixed bundle.

\(^{25}\)The Newton Polytope of a valuation is the convex hull of its domain.

\(^{26}\)The construction uses Legendre-Young duality: see Murota (2003, especially Chap. 9) but is not a precise duality: as we will see, information is lost, so that a single SNP corresponds to a set of THs.
of that price. Thus, they are the bundles, \( x \), for which \((x, u(x)) \) lies on the minimal concave function which is everywhere weakly greater than \( u \) (in this example, they are every bundle except \((1, 1)\)). We extend this latter function to the convex hull of the set of feasible bundles, and call its graph the \textit{roof} of the valuation. Looking at Figs. 2b and 2c shows that the roof is simply the ‘top’ of the convex hull of the points \((x, u(x))\) with respect to the final coordinate.

Because the agent is indifferent between several bundles at some prices, the roof is composed of pieces of hyperplanes that meet along lower-dimensional linear pieces; like the TH, it is a ‘polyhedral complex’. (In our 2-good example, the roof includes pieces of planes, line segments, and points.) Each of the roof’s vertices is at a bundle which is the unique demand for some prices. Conversely, a top-dimensional cell of the roof (a piece of hyperplane) corresponds to a set of bundles between which the agent is indifferent at some unique price, \( p \). The vector \((-p, 1)\) is normal to the hyperplane in question (i.e., the one which contains the bundles, \( x \), maximising \((-p, 1). (x, u(x)) = u(x) - px\)). More generally, any cell of the roof is the intersection of some tangent hyperplane(s) with the roof, and so is the demand set for some price(s).

By projecting downwards the pieces of hyperplanes that form the roof, we can subdivide the convex hull of the set of feasible bundles–see Fig. 2c. Since the convex hull is just a ‘Newton Polytope’, we call the resulting object a \textit{Subdivided Newton Polytope} (SNP). We call the projection of a vertex of the roof a \textit{vertex} of the SNP (which therefore corresponds to a bundle that is uniquely demanded at some price), and call the projection of a line segment of the roof an \textit{edge} (which therefore corresponds to the line joining two uniquely-demanded bundles).

Note in particular, therefore, that an edge of the SNP indicates the existence of prices, \( p \), at which the agent is indifferent between the bundles, \( x \) and \( x' \), corresponding to the ends of the edge. Moreover, these prices form a facet of the TH, because a line segment of the roof has an \((n - 1)\)-dimensional family of tangent hyperplanes passing through it, and so an \((n - 1)\)-dimensional space of vectors \((-p, 1)\) normal to it, for prices \( p \) for all of which both \( x \) and \( x' \) are demanded. So since, as we noted in Section 2.2, \( u(x) - px = u(x') - px' \), that is, \( p.(x' - x) = \text{constant} \), for all these price vectors, \( p \),
each (1-dimensional) edge of the SNP is normal to the ((n − 1)-dimensional) facet that corresponds to it in the TH.

More generally, each k-cell of the SNP is the convex hull of the bundles which form the demand set for some price; and the set of prices p for which these bundles are contained in the demand set $D_u(p)$ is an (n − k)-cell of the TH that is orthogonal to it:

**Lemma 2.8.** There is a 1-1 correspondence between the k-cells of the SNP and the (n − k)-cells of the TH. Each TH cell is orthogonal to its corresponding SNP cell. That is, $(p' - p) \cdot (x' - x) = 0$, for all $p, p'$ in the TH cell and $x, x'$ in the SNP cell.

In particular, each edge of the SNP is normal to its corresponding facet in the TH.

The SNP and TH of the valuation of Fig. 2a are pictured in Figs. 3a and 3b, shaded to correspond both to the picture of the roof (Fig. 2c) and to each other, so that dual geometric objects are shaded the same way.

![Diagram](image)

Figure 3: (a) The SNP. (b) The TH of the valuation given in Fig. 2a, and so corresponding to the SNP in (a). (c) Another TH corresponding to the SNP in (a), but with a different valuation.

Thus the 0-cells (vertices) of the TH at the prices (4,6), (2,2), and (1,1) correspond to the dotted-, wavy-, and light-grey-, shaded 2-cells (areas) of the SNP, respectively; each of the seven UDRs of the TH corresponds to one of the seven bundles shown as white circles in the SNP.\(^\text{28}\)

Notice that the (dark) grey horizontal edge at the top of the SNP passes through the grey bundle. This edge is twice the length of the primitive integer vector, (1,0), in its direction, so we say that the edge has *length 2*;\(^\text{29}\) it is dual to the (dark) grey vertical facet of the TH which correspondingly has weight 2, and is so labelled. (Recall from\(^\text{27}\) we typically draw a SNP without axes, since replacing $A$ with $A + c$ for some $c \in \mathbb{Z}^n$ and re-defining $u$ to correspond gives us the same SNP and TH.

\(^{28}\)Clockwise from the top right of the TH, the bundles demanded in the UDRs are (0,0), (0,1), (0,2), (1,2), (2,2), (2,1), and (2,0).

\(^{29}\)Of course, this length is not Euclidean length; all the other edges of this SNP have length 1.
Section 2.2 that a facet’s ‘weight’ times its primitive integer normal vector is the change in demand between the UDRs it separates.)

Neither the grey bundle, nor the black bundle, is a vertex of the SNP. So neither bundle is *uniquely* demanded for any price. Furthermore, *neither* the TH nor the SNP tells us whether a non-vertex bundle such as one of these is demanded at all. If the valuation is affine in the relevant range, the bundle is in the roof (although not a vertex of it), and so is demanded. But if the valuation is non-concave at the non-vertex bundle, this bundle’s value lies strictly below the roof, i.e., \((x, u(x))\) is not in the roof for this bundle, \(x\). In such a case the bundle is “jumped over” as we cross between UDRs.

The grey bundle is an example of the former case. Its valuation, 4, is precisely the average of the valuations (0 and 8) of the bundles \((0,0)\) and \((2,0)\) (see Figs. 2a and 2c). It is therefore demanded for some prices (here, \((4,p_2)\) for \(p_2 \geq 6\)–see the TH) but is not the unique demand at any price.

However, the black bundle’s value is strictly below the level of the “roof” (see Fig. 2c), so it is never demanded at any price.\(^{30}\)

Note that because the central wavy-shaded (five-sided) SNP cell is the only SNP cell that the black bundle lies in, the corresponding wavy-shaded 0-cell (at price \((2,2)\)) of the TH in which it is currently ‘hidden’, was the only price at which this bundle might have been demanded: when a bundle is demanded, its position in the SNP dictates at which price(s). That is:

**Lemma 2.9.** For \(x \in A\), exactly one of the following holds:

1. \(x\) is not demanded for any price;
2. \(x\) is in an SNP cell iff, for every \(p\) in the corresponding TH cell, \(x \in D_u(p)\).

The results of this section are laid out formally in Appendix A.1.3

### 2.5 The correspondence between sets of valuations and SNPs

It is easy to see that multiple valuation functions, \(u(x)\), yield the same SNP. That is, there are many ways of changing the values of the bundles that do not affect which vertices are directly connected to which, in the roof of the agent’s valuation: they do not affect which sets of bundles can jointly form the demand set. (For example, consider how we can change the valuation of Fig. 2a without changing the connections in Fig. 2c.) However, such changes cannot affect the correspondences and orthogonality relationships discussed in the previous subsection between the cells of the SNP and of the TH.

So a single SNP corresponds to a set of THs, all of which have cells with the same dimensions and slopes, connecting to one another in the same way. We say that such a set of THs (and the set of corresponding valuation functions) are all of the same combinatorial type; they all correspond to agents who make the same trade-offs between additional units of goods, even if not always at the same prices.

\(^{30}\)But if its valuation were greater, so the corresponding bar in Figs. 2b and 2c just touched the roof, then it would still not be a vertex, but it would be demanded at the price corresponding to the wavy-shaded 0-cell (that is, \((8,8)\)) that is a vertex of the TH. And if it had an (even) higher valuation (so “poked through” the current roof), then the corresponding SNP point would become a vertex, and the corresponding TH 0-cell would “open up” to form new a UDR corresponding to the range of prices at which the bundle \((1,1)\) would then be demanded.
In sum,

**Theorem 2.10** (Mikhalkin, 2004, Proposition 2.1.). *There is a 1-1 correspondence between SNPs of THs and combinatorial types of THs.*

Fig. 3c gives a TH of another valuation that has the same combinatorial type as the valuation of Fig. 2a—its SNP is therefore also that shown in Fig. 3a.

It should be clear from our example that, starting from any SNP, it is easy to find the combinatorial type of the TH; the exact location of the TH for any specific valuation function can then easily be worked out from the values of the different bundles.\(^{31}\) Conversely, given any TH, it is easy to determine which bundle is demanded in each UDR starting from the demand in any one UDR (see discussion above Definition 2.1), and then also easy to find the corresponding SNP.\(^{32}\)

Furthermore, if the set of feasible bundles is not too large, it is easy to list all the possible SNPs, and so also all the possible combinatorial types of THs, that is, every possible distinct structure of trade-offs that an agent might make between the goods. Figs. 9 and 10 in Appendix A.1.4 give examples.

### 2.6 Representation in Price Space vs. Representation in Quantity Space

Although the TH and SNP are ‘dual’, the price and quantity representations have different properties and are useful in different contexts.

We will see (in Section 3) that the relationship between the economic properties of a valuation and the geometric properties of its TH in price space allows us to classify valuations into “demand types”, such as substitutes, complements, etc. It would be almost equivalent to categorise valuations using the SNP in quantity space. However, an important distinction is that any geometric object satisfying the simple ‘balancing condition’ of Definition 2.2 is the TH of some valuation (see Theorem 2.3), but *not* every subdivision of every Newton polytope arises from some valuation. Moreover, there seems to be no simple check of whether or not a given subdivision corresponds to any valuation function.\(^{33}\)

So it seems easier to specify all the geometric objects in price space that represent a particular economic property, than to do this in quantity space where we have to take care to restrict attention to cases that can actually arise. And Theorem 2.3 guarantees that if we develop examples to, e.g., test conjectures, working in price space, then the corresponding valuations will exist; we have found in practice that this is considerably easier than developing valuation functions directly.

\(^{31}\)For example, for the valuation of Fig. 2a, it is clear from the valuations of bundles \((1,0)\) and \((0,1)\) that the dotted-shaded 0-cell of the TH is at \(p = (4,6)\), since 4 and 6 are the prices below which the agent will first buy any of goods 1 and 2, respectively, when the other good’s price is very high. And the coordinates of the wavy-shaded 0-cell must be \((2,2)\) since \(8-6=2\) is the incremental value of a second unit of good 2, when the agent has no unit of good 1, and \(10-8=2\) would be the incremental value from then adding a unit of good 1, etc.

\(^{32}\)In two dimensions, we know each UDR (area) in the TH corresponds to a vertex (point) in the SNP. A facet (line-segment) in the TH corresponds to an edge in the SNP in the orthogonal direction, joining the vertices corresponding to the UDRs on either side; its length is given by the weight of the facet. So we can immediately draw all the vertices and edges.

\(^{33}\)Gathmann (2006, Fig. 7) shows a subdivision that corresponds to no valuation function.
Working in price space also makes it much easier to aggregate agents’ valuations (see Section 3.3).

Furthermore, while an SNP shows only the collections of bundles among which the agent is indifferent for some price vectors, THs show clearly which bundles are demanded in which regions of prices. Thus THs are often easier to interpret.

So we will mostly develop our ideas in price space.

However, the different perspective offered by the geometric objects in quantity space is also valuable. In particular, some essential information that is only implicit in the TH becomes obvious in the SNP. For example, we will see in Sections 4 and 5 that a 0-dimensional, or low-dimensional, cell of the TH sometimes “hides” important detail that is much more easily seen and interpreted in the higher-dimensional dual object in the SNP.

Another virtue of the SNP is that the easiest way to compute the THs of specific valuations is often via first computing the SNPs, so even when the TH is an easier-to-understand representation, the SNP helps us construct it more quickly.34

The fact that the different representations are useful in different contexts makes the ability to move easily between them, using duality, especially valuable.

3 “Demand Types”

3.1 Defining “demand types”

The previous section suggests classifying valuations according to the vectors that are normal to their THs’ facets:

Definition 3.1. A valuation is of demand type \( D \) if all the primitive integer normals to the facets of its associated TH lie in a set, \( D \), of primitive integer vectors in \( \mathbb{Z}^n \), such that if \( v \in D \) then \(-v \in D\).35

For example, the valuation of Fig. 1 is of demand type \( \pm\{(1, 0), (0, 1), (-1, 1)\} \) as, of course, are many other valuations, for example, all those shown in Figs. 8a-c. Note that a valuation has any demand type which contains the facet normals of its TH; we do not restrict to the minimal such set.36

Recalling Lemma 2.8, we can equivalently classify valuations according to the vectors in the directions of their SNPs’ edges.37 Recall also, however, that all THs correspond to valuations of the demand type that their facet normals’ vectors define, but not all

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34For example, in constructing the TH of the valuation of Fig. 2a, going via the SNP (see Section 2.4 and note 31) both separates the question “in what directions are there line segments?” from the question “where in space are they?”, and also clarifies which bundles have to be compared with which.

35We will write “demand type \( D \)” for the set of valuations defined by the set \( D \) of vectors; given a demand type, we will refer to the defining vectors as the “demand type’s vectors” for the set of vectors defining a demand type. Note our definition does not consider the weights on facets; see Baldwin and Klemperer (2012, note 25, and 2014, note 42).

36For example, the valuations of Figs. 1 and 8a-c are also of demand type \( \pm\{(1, 0), (0, 1), (-1, 1), (-2, 1)\} \) which is the minimal demand type of the valuations of Figs. 2–3.

37Danilov, Koshevoy and their co-authors examine these vectors in quantity space in the course of their impressive body of work that, we will see in Section 4.1, has close connections to ours (see Danilov et al., 2001, and Danilov et al., 2003, 2008, 2013). However, they do not use them to create a taxonomy.
subdivisions of Newton polytopes whose edges are among the vectors of a demand type correspond to (any) valuations.

3.2 Comparative Statics, and Substitutes, Complements, etc.

Since the vectors defining a demand type are the set of all the possible directions of the TH’s facet normals, and since these in turn specify the possible directions of demand changes as we cross the facets between UDRs, combinations of these vectors specify all the possible changes in demand between prices in UDRs. Since the UDRs are dense in price space, these are the possible changes in demands that can generically result from a small change in prices.38

It follows straightforwardly that demand types provide simple characterisations of concepts such as substitutes and complements.

For substitutes, an increase in a good’s price, between prices at which demand is unique, might decrease but cannot increase the demand for that good, and cannot result in the agent decreasing its demand for other goods. (See, for example, Figs. 1 and 3b-3c, for this property holding, and Fig. 4 for it failing.) So the vectors that are normal to a facet may have two non-zero entries of opposite signs, but cannot have two non-zero entries of the same sign (see Appendix A.2.1 for details).

Definition 3.2. A valuation \( u \) is ordinary substitutes\(^{39} \) if, for any prices in UDRs such that \( p' \geq p \), if \( D_u(p) = \{x\} \) and \( D_u(p') = \{x'\} \), we have \( x'_k \geq x_k \) for all \( k \) such that \( p_k = p'_k \).

Proposition 3.3. A valuation is of a demand type whose vectors each have at most one positive and at most one negative coordinate entry if it is an ordinary substitutes valuation.

Similarly, for complements, if any good’s price increases, then the agent may reduce, but cannot increase her demand for other goods, so there is no facet whose normal vector has two non-zero entries of different signs:

Definition 3.4. A valuation \( u \) is ordinary complements if, for any prices in UDRs such that \( p' \geq p \), if \( D_u(p) = \{x\} \) and \( D_u(p') = \{x'\} \), we have \( x'_k \leq x_k \) for all \( k \) such that \( p_k = p'_k \).

of demand—we, by contrast, develop a general framework to understand them in economic terms (see also Baldwin and Klemperer, 2012, 2014 and in preparation-b). In particular, as Danilov et al. work almost exclusively in quantity space, they do not see these vectors as giving changes in demand as we move around in price space.

38See Baldwin and Klemperer (2014, in preparation-b) for full discussion of possible demand changes to and from prices at which demand is non-unique.

39We call “ordinary substitutes” what most others (e.g., Ausubel and Milgrom, 2002) simply call “substitutes”. We do this for clarity, since some have defined “substitutes” in other ways. In particular, although Kelso and Crawford’s (1982) definition is equivalent in their model, it is not generally equivalent if it is extended to multiple units of three or more goods (see Danilov et al., 2003, Ex. 6 and Thm 1). Our definition (3.2) seems the most natural one in the general case. It is also equivalent to several properties that seem to naturally characterise “substitutes”, and to the indirect utility function \( \max_{x \in \mathcal{A}} (u(x) - p \cdot x) \) being submodular—see Baldwin, Klemperer and Milgrom (in preparation). See also Baldwin and Klemperer (2014). Hatfield et al. (2013, see Section 6.1) and Danilov et al. (2003) use equivalent definitions to this definition, and the latter authors make a similar observation to our next proposition (3.3) when they say ‘each cell of a valuation’s parquet is a polymatroid’.

15
Figure 4: A facet (shaded) with its normal (the arrow shown in bold). Increasing either $p_1$ (as shown with a dotted arrow), or $p_3$, demonstrates complementarities between goods 1 and 3, as the bundle demanded switches from (1,0,1) to (0,1,0).

**Proposition 3.5.** A valuation is of a demand type whose vectors’ non-zero coordinate entries are all of the same sign iff it is an ordinary complements valuation.

Note that, by contrast with the standard definitions, 3.2 and 3.4, our way of classifying demand “types” clearly demonstrates both the lack of symmetry between substitutes and complements, and the reason for it: the substitutes demand type includes only vectors for which each pair of non-zero entries are of opposite signs, while the complements demand type includes only vectors for which each pair of non-zero entries have the same signs—this implies that in more than two dimensions complements permits vectors with any number of non-zero entries, whereas substitutes permits at most two non-zero entries.

The reason for the asymmetry is that, if any one good can trade-off against two others at the same price, the two other goods must be complementary. Consider, for example, Fig. 4, which illustrates a facet with normal $\mathbf{n} = (1,-1,1)$, defined by $\{ \mathbf{p} \in \mathbb{R}^3 \mid p_1 + p_3 = p_2; \ p_1, p_2, p_3 \geq 0 \}$. Moving from the UDR with $p_1 + p_3 < p_2$ (“behind” the facet) to the UDR with $p_1 + p_3 > p_2$ (“in front of” the facet), by increasing the price of either good 1 or good 3, changes the bundle demanded from (1,0,1) to (0,1,0) and so reduces demand for both goods 1 and 3.40

It is easy to find the demand types that correspond to other standard classes of valuations. For example,

40So even when all goods are mutual substitutes there can never be trade-offs between more than two of them across a single facet.

To illustrate why the conditions for indivisible goods to be substitutes are so restrictive, consider a consumer who regularly makes three kinds of trips: journey A can be made only by bus or train; journey B can be made only by car or train; journey C can be made only by car or bus. Thought of as divisible goods, the three modes of transport are all mutual substitutes. But if the price of either bus tickets or train tickets is slightly raised, a consumer might buy a car and reduce her use of both forms of public transport, which are therefore locally complements—that is, the car takes the role of good 2 in the situation pictured in Fig. 4.

16
Proposition 3.6. A valuation is strong substitutes if it is concave and is of a demand type whose vectors each have at most one +1 entry, at most one -1 entry, and no other non-zero entries.

Note that with \( n \) distinct kinds of good, a strong substitutes demand type contains at most \( n(n + 1)/2 \) vectors and their negations.

We discuss some other demand types of interest in Sections 6.3-6.4.

3.3 Aggregate Demand, and the “Demand Type” of an Aggregate Valuation of Multiple Agents

An important feature of our “demand types” classification—that, in particular, greatly facilitates the study of equilibrium—is that the demand type of the aggregate valuation of multiple agents is just the union of the sets of vectors that form the individual agents’ demand types.

We now consider a finite set of agents, \( j = 1, \ldots, m \): agent \( j \) has valuation \( u^j \) for integer bundles in a finite set, \( A_j \). It is obvious that the agents’ aggregate demand, \( D_U(p) \), at any price \( p \) is simply the sum of their individual demands at that price, that is, 
\[
D_U(p) = \left\{ \sum_j x^j \mid x^j \in D_{u^j}(p) \right\}.
\]
So it is also clear that the superimposition of the individual agents’ THs is the TH of an “aggregate valuation” that corresponds to the aggregate demand. More precisely, let \( \mathcal{T}_U \) be the union of the individual THs, \( \mathcal{T}_{u^j} \), with facet weights given by adding the weights of facets that coincide. \( \mathcal{T}_U \) is clearly balanced since the individual THs are, and so we can apply Theorem 2.3 to see that there exists a quasilinear valuation, \( U \) corresponding to \( \mathcal{T}_U \). And we can confirm our interpretation by observing that \( \mathcal{T}_U \) implies aggregate demand at a price is unique iff all the individual demands are, and the change in aggregate demand between any prices is just the sum of the changes in the individual demands as we cross facets of the individual THs. (We give the precise form of \( U \) below, but it is cumbersome to work with and we seldom do so).

Fig. 5 illustrates this by showing the THs of two simple valuations, with domain \( \{0, 1\}^2 \): a substitutes valuation, \( u^s(x_1, x_2) = 1 \) if \( x_1 \geq 1 \) or \( x_2 \geq 1 \), \( u^s(x_1, x_2) = 0 \) otherwise (Fig. 5a); a complements valuation, \( u^c(x_1, x_2) = 1 \) if \( x_1 \geq 1 \) and \( x_2 \geq 1 \), \( u^c(x_1, x_2) = 0 \) otherwise (Fig. 5b); and of the aggregate of these two valuations (Fig. 5c).

Note that when cell interiors from different agents intersect, the cells are split up into new, smaller cells in the aggregate TH, with a new, lower-dimensional, cell at their intersection. For example, in Fig. 5c, the point \((\frac{1}{2}, \frac{1}{2})\) is a 0-cell, on the boundary of four distinct 1-cells.

It is now immediate that demand ‘type’ is preserved under aggregation:

\[\text{[Note: Footnote 41: See Baldwin and Klemperer (2014, Corollary 5.20, and in preparation-b). Milgrom and Strulovici (2009) define valuations to be ‘strong substitutes’ if every unit of every good is a substitute for every other unit of every good, in the sense of Kelso and Crawford (1982). For other equivalent definitions see Milgrom and Strulovici (2009), Baldwin and Klemperer (2014) and Baldwin, Klemperer and Milgrom (in preparation). In particular, Danilov et al. (2003, Proposition 7) show valuations are their “step-wise gross substitutes” if they are both concave and (in our language) having the edges of its SNP in the set we describe. (We use Milgrom and Strulovici’s later terminology because it seems to have become more standard). Figs. 1, 5a, and 8a-c show examples of THs of “strong substitutes” valuations.]}\]
Figure 5: The THs of (a) a simple substitutes valuation; (b) a simple complements valuation; (c) the aggregate of the simple substitutes and simple complements valuations shown; (d) the aggregate of the simple complements valuation shown and a simple substitutes valuation with lower values for each unit.

Figure 6: SNPs corresponding to the THs shown in Fig. 5.

**Proposition 3.7.** Valuations $u^j$ are of demand type $\mathcal{D}$ for $j = 1, \ldots, m$ iff the aggregate TH, $T_U$, is of demand type $\mathcal{D}$.

The ability to “add” THs straightforwardly is a merit of working in price space for these purposes.

Working in quantity space would be possible, using the standard result—see Appendix A.2.2—that, since agents’ preferences are quasilinear, $D_U(p)$ is just what would be demanded by a single agent whose valuation, $U(y)$, is the greatest sum of the valuations, $u^j$, that can be attained by dividing the bundle, $y$, between the agents, that is, $U(y) = \max \left\{ \sum_j u^j(x^j) \mid x^j \in A_j, \sum_{j=1}^m x^j = y \right\}$. But, because finding any value of $U(y)$ requires considering all possible partitions of $y$ among the agents, which is both time-consuming and unintuitive, doing this in quantity space is harder.

We also cannot find the SNP of an aggregate valuation from the individual SNPs, in quantity space, since neither the aggregate valuation, nor its combinatorial type, is uniquely defined by the combinatorial types of the individual valuations (that is, by the individual SNPs). However, working in price space, starting with specific THs, we can easily find the aggregate TH, and hence the aggregate SNP (and other information about aggregate demand) for the specific case.

For example, it is easy to see that if we alter the substitutes valuation of Fig. 5a
above, to $u^\ast(x_1, x_2) = 1/4$ if $x_1 \geq 1$ or $x_2 \geq 1$, $u^s(x_1, x_2) = 0$ otherwise, its TH is of the same combinatorial type as before. However, the 0-cell that was at $(1, 1)$ has moved to $(1/3; 1/4)$, so the TH of the aggregate, $U'$, of it with the complements valuation, $u^c$, of Fig. 5b is that shown in Fig. 5d. And it is also straightforward that the SNP of both $u^s$ and $u^\ast$ is that of Fig. 6a, and that the SNP of $u^c$ is that of Fig. 6b. However, the SNP of the aggregate valuation $U$ of $u^s$ and $u^c$ is that of Fig. 6c, while the SNP of the aggregate valuation $U'$ of $u^\ast$ and $u^c$ is that of Fig. 6d. Clearly, there is no unique aggregate SNP corresponding to the SNPs of Fig. 6a and Fig. 6b.

4 The Existence of Competitive Equilibrium for a Demand Type

A beautiful aspect of our “demand types” classification is that it leads us naturally to a powerful theorem giving a necessary and sufficient condition for the existence of competitive equilibrium. This theorem requires much weaker assumptions about agents’ preferences than used in the existing leading economics literature, so our condition for equilibrium is correspondingly much more general. It immediately generalises, for example, the equilibrium results of Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). In particular it is not necessary for all agents to have “strong substitutes” valuations (or some basis change thereof) for equilibrium to always exist; complements valuations guaranteeing equilibrium are easy to find. Instead, concavity and a “unimodularity” condition explained below are all that are required.

Concavity The crucial role of concavity is that, since concave functions are precisely those for which every possible bundle is demanded at at some price (Lemma 2.5), there exists a competitive equilibrium price vector for every possible market supply iff the agents’ aggregate valuation is everywhere concave. And, although each individual agent’s valuation being concave is not sufficient for their aggregate valuation to be concave, our geometric approach shows us a simple condition that is sufficient.

Unimodularity Unimodularity is important in understanding the SNP’s geometry.

Definition 4.1. A set of vectors in $\mathbb{Z}^n$ is unimodular if every linearly independent subset can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant $\pm 1$.

By “the determinant” of $n$ vectors we mean the determinant of the $n \times n$ matrix which has them as its columns.\footnote{We ignore the order of the vectors since we are only ever interested in the absolute values of determinants.}

To understand the geometric importance of unimodularity, note that a set of $n$ linearly independent integer vectors are the edges of an $n$-dimensional parallelepiped. This shape contains no integer point (either in its boundary or in its interior) aside from its vertices iff its volume is 1. But this volume is just the absolute value of the determinant of the vectors along its edges.
Moreover, if this volume is 1 then it follows that any lower-dimensional parallelepiped spanned by a subset of these vectors also contains no integer point other than its vertices. So if unimodularity is satisfied, this critical property must hold. Unimodularity is also necessary for this property (see Remark A.15). And this property implies that we can move between any integer bundles in the linear span of this parallelepiped by taking integer combinations of these vectors: the ‘integer lattice’ in this linear span is made up of repeated copies of the parallelepiped. That is, the vectors are an ‘integer basis’ for this subspace.

If the set of vectors spans \( \mathbb{R}^n \), then there exist sets of \( n \) of them that are linearly independent; it is therefore, of course, sufficient to check that all \( n \)-element sets have determinant \( \pm 1 \) or 0.

Alternatively equivalent conditions for unimodularity are given in Remark A.15.

Importantly, therefore, unimodularity of a set \( \mathcal{D} \) defining a demand type is not too hard to check, and we refer to ‘unimodular demand types’ in the obvious way.

### 4.1 Necessary and Sufficient Condition for Equilibrium to always Exist for a Demand Type

Since, of course, an individual agent with a non-concave valuation function fails to always have a competitive equilibrium, we now only consider concave individual valuations. Similarly, we only consider bundles in the domain of the aggregate valuation, since a bundle clearly cannot be demanded if it is outside the range of consideration by the agents. For brevity, we will refer to the domain of aggregate valuation as “the domain”:

**Theorem 4.2.** A competitive equilibrium exists for every set of agents with concave valuations of demand type \( \mathcal{D} \) and any supply bundle in the domain iff \( \mathcal{D} \) is unimodular.

In particular, it is immediate from the discussion in the previous subsection that

**Corollary 4.3.** With \( n \) goods, if the vectors of \( \mathcal{D} \) span \( \mathbb{R}^n \), then a competitive equilibrium exists for every set of agents with concave valuations of demand type \( \mathcal{D} \) and any supply bundle in the domain iff every subset of \( n \) vectors from \( \mathcal{D} \) has determinant 0 or \( \pm 1 \).

We sketch the proof and intuition for these results in the next subsection (4.2). Full details are in the Appendix. A remarkable series of papers by Danilov, Koshevoy and their co-authors, has developed results that are very closely related to ours. In particular, Theorems 1, 3 and 4 of Danilov et al. (2001) together provide a sufficient condition for equilibrium, which is analogous to our condition on demand types.\(^{43}\) However, the interpretation or useful-

\(^{43}\)Their sufficient condition for a class of valuations to have equilibrium for any supply is that the valuations be “\( \mathcal{D} \)-concave” for some “class of discrete convexity” \( \mathcal{D} \). Here, “\( \mathcal{D} \)-concave” valuations are concave valuations such that every demand set \( D_u(p) \) belongs to the set “\( \mathcal{D} \)” of subsets of \( \mathbb{Z}^n \). A “class of discrete convexity” is a collection of sets such that every set is discrete convex, and every Minkowski sum and every Minkowski difference of the sets is discrete convex. They also show that \( \mathcal{D} \) is a class of discrete convexity if the edges of the convex hulls of the sets in \( \mathcal{D} \) form a unimodular set of vectors. The proof of this, their Theorem 4, is given by Danilov and Koshevoy (2004, Thm. 2). Note that Danilov et al.’s use of the notation \( \mathcal{D} \) is not connected with our use of \( \mathcal{D} \) to represent demand types.
necessity of their result is not made clear; by contrast, our theorem both demonstrates the applicability of the result, and clarifies the connections to existing economic results.\textsuperscript{44}

Danilov et al. also prove no necessity result. Because they have not developed their definition as a taxonomy of demand, in the way we do with demand types, they do not show the necessity of unimodularity for the existence of competitive equilibrium. Once our concept of demand types is introduced, however, a necessity result can easily be developed.\textsuperscript{45}

Danilov et al. moreover state their results under different assumptions from ours. They assume the domain, \( A \), of every agent’s valuation is \( \mathbb{Z}_{\geq 0} \), which precludes, for example, the application to agents who both buy and sell which our more general assumption permits.\textsuperscript{46}

Finally, although the techniques we use to prove our results are novel, they seem simpler and more accessible to economists than Danilov et al.’s very advanced mathematical techniques. So we will prove the theorem using our alternative method, which understands the result as an application of “intersection multiplicities” in tropical geometry.\textsuperscript{47}

### 4.2 Intuition and Sketch of Proof for Theorem 4.2

#### 4.2.1 The Role of Intersections

The first insight is that we can determine whether equilibrium exists by focusing on the intersection of individual THs: we know equilibrium always exists, that is, every bundle is demanded at some price, \( \text{iff} \) the aggregate valuation is concave \( \text{iff} \) the aggregate demand set is discrete-convex at every price (Lemma 2.5). But if all but one of the agents have unique demand at some price, the aggregate demand set is simply the

\textsuperscript{44}We will see that Theorem 4.2 generalises the results on equilibrium in work subsequent to Danilov et al.‘s, including in Hatfield and Milgrom (2005), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), and Teytelboym (2014). The absence in Danilov et. al’s work of our notion of demand types or of any economic interpretation of their concept of “\( D \)-concavity”, and the presentation of their work in relatively unfamiliar terms (namely the relationships between sets of primitive integer vectors which are parallel to edges of specific collections of integral pointed polyhedra and the “classes of discrete convexity” that they define) seems to have resulted in leading economists being unaware of their work or of its implications. (We were also unaware of their work until after we had developed our own results.)

\textsuperscript{45}The sufficiency part of our theorem follows from combining Theorems 1, 3 and 4 of Danilov et al. (2001). To understand the relationship between these theorems and our Theorem 4.2, observe that in their Theorem 4 certain sets of “primitive integer vectors, which are parallel to edges of” a certain “collection of integral pointed polyhedra” are analogous to our demand types; furthermore, the “classes of discrete convexity” they define are analogous to a set of demand sets \( D_u(p) \) which are all discrete-convex and such that this property is preserved under aggregation. It is not hard to also show, using our Lemma 2.9, the necessity of a demand type giving rise to a “class of discrete convexity” for competitive equilibrium to always exist, and in this way we can also derive our necessity result from their work.

\textsuperscript{46}For example, our model, unlike theirs, applies to (and extends) Hatfield et al. (2013)–see Section 6.1. In fact Danilov et al.’s assumption seems unnecessary for them, so we could develop our full theorem by extending their work. See our Note 45, above. See also our discussion about the distinction between their approach and ours in Note 37. Their work also covers some of the examples in Sections 4.3.3, 4.3.4 and 6.3, as we note in those sections.

\textsuperscript{47}It was this theory that inspired our (independent) development of our results. Full details of our proof are in Appendix A.3.1.
shift of the remaining agent’s demand set by the other agents’ (unique) demands. And this set must be discrete-convex, since we assumed that every individual valuation is concave. So we only need to check prices at which two or more agents have non-unique demand. That is:

**Lemma 4.4.** Equilibrium exists for every supply bundle in the domain iff the aggregate demand set is discrete-convex at the intersection of agents’ THs.

### 4.2.2 Necessity

Consider, therefore, a price at which two or more agents’ TH facets of weight 1 intersect (and other agents have unique demand). The corresponding cell in the aggregate SNP is then a parallelepiped whose edges are the normals to those intersecting facets. So these edges are vectors of the demand type of the agents’ valuations.

If the demand type is not unimodular then, as discussed earlier in Section 4, we can find a set of its vectors for which such a parallelepiped contains integer point(s) that are not its vertices. And we saw in Section 2.4 that bundles that are not SNP vertices are “hidden” inside the corresponding cells of the corresponding TH, and may not be demanded at the corresponding prices. Indeed, in this case, each of the relevant agents has just two bundles in its demand set, so with $s$ agents there are only $2^s$ different possible aggregate demands, and these must correspond to the parallelepiped’s $2^s$ different integer vertices. The non-vertex bundle(s) therefore cannot be demanded at the corresponding price. So the aggregate demand set is not discrete-convex at this price, and competitive equilibrium therefore fails if the supply is such a non-vertex bundle (see Lemma 2.9).

This logic shows necessity (see Lemma A.16 in Appendix A.3.1). It also demonstrates how to easily construct examples of failure of equilibrium for any non-unimodular demand type.

### 4.2.3 Sufficiency for Simple Cases

If THs intersect only in the simple form just discussed, a parallel argument to the one above demonstrates sufficiency: if the demand type is unimodular, then no parallelepiped corresponding to an intersection price contains a non-vertex integer point, so no bundles are “hidden” in the intersection and the aggregate demand set is discrete-convex at every price, implying our sufficiency result.

The importance of unimodularity will carry over to general intersections. The broader intuition is that unimodularity of the set of facet normals means we can reach all bundles by taking integer combinations of this set of vectors. That is, all the bundles are connected by edges in quantity space and, correspondingly, all aggregate demands can be achieved by crossing appropriate facets in price space.

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48 We assume that only facets (and no lower-dimensional cells in the individual THs) intersect at this price. This scenario is generic when $n = 2$ but not for $n \geq 3$.

49 Unimodularity is equivalent to the tropical intersection multiplicity being equal to one in such a case (see Section 5 and Appendix A.4.).

50 Because we specified the facets all had weight 1, there are necessarily just 2 bundles in each demand set. If any facet had a greater weight, integer points that are not vertices will be “weakly-demanded” (i.e., demanded, but never uniquely demanded), like the “grey” bundle, $(1,0)$ of Fig. 3, that we discussed at the end of Section 2.4; the SNP cell will consist of consecutive copies of the parallelepiped described here.
4.2.4 Sufficiency when the TH Intersection is “transverse”

To develop a general proof for sufficiency, we begin by focusing on “transverse” TH intersections:

**Definition 4.5.** THs $T_u^1$ and $T_u^2$ intersect transversally at $p$ if $\dim(C^1 + C^2) = n$, in which $C^i$ is the minimal cell of $T_u^i$ containing $p$, for $i = 1, 2$, and $C^1 + C^2$ is the set-wise (Minkowski) sum of these cells.

The intersection of THs $T_u^1$ and $T_u^2$ is transverse if they intersect transversally at every point of their intersection.

THs $T_u^1, \ldots, T_u^k$ intersect transversally at $p$ if $T_u^j+1$ intersects the TH of the aggregate of the valuations $u^1, \ldots, u^j$ transversally at $p$, for all $j = 1, \ldots, k - 1$.  

Thus transverse intersections are generic intersections, as we make precise below (see Proposition 4.6). For example, in two dimensions, two lines crossing at a single point are intersecting transversally, but two coincident lines are not, and nor are three lines crossing at a single point (since if there are just two intersecting THs, the one that contains two of the lines has a 0-cell at this point, and if we are considering three THs, the aggregate of the first two has a 0-cell at this point). In three dimensions, a line meeting a plane in a single point is a transverse intersection (point), as is two planes meeting in a line, or three planes meeting in a single point.

The important point about transverse intersection prices is that the changes in bundles considered by different agents at any such price are linearly independent—that is, the spaces of the possible changes in the individual agents’ bundles have zero intersection (Lemma A.25 makes this point precise). This means that there is only one possible way to apportion a change in the aggregate supply bundle among the different agents.

To show that a supply bundle, that would be demanded at a transverse intersection price if it were demanded anywhere, is in fact demanded, we start from any supply bundle that is demanded in an adjacent UDR to the intersection. We then consider two different ways of thinking about dividing up the change between the two supply bundles among the agents; the fact that these two divisions must be the same will show that the supply bundle in question is also demanded.

The first way we think about dividing up the change in aggregate supply observes that the SNP cell in the aggregate SNP is the Minkowski sum of the individual SNP cells. So any change in total supply within the aggregate SNP cell must be decomposable as a sum of individual changes, each of which is within an individual SNP cell and therefore within the convex hull of an individual demand. So we have assigned each agent a bundle in the convex hull of its demand set at this price, although we have not yet demonstrated that these new bundles are integer bundles.

For example, when two agent’s THs intersect transversally in two dimensions, the corresponding cells in the aggregate SNP are just parallelepipeds (i.e., parallelograms) of the kind discussed above, so in each case the aggregate change can be broken down into parts along the edges of this parallelogram, and each agent can be allocated the additional supply corresponding to “its” edge of the parallelogram. (In more than two dimensions, the geometry is a little more complicated, because the cells in the aggregate SNP that correspond to transverse intersection prices need not be parallelepipeds.)

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51It is straightforward that this definition is independent of the order in which the THs are taken.
The second way we think about dividing up the change in aggregate supply uses the unimodularity of $D$: we can fix a basis for each individual’s change in demand at the intersection price, made up of edges of that individual’s SNP cell (equivalently, its facet normals at the price). Transversality means that taking all these bases together creates a basis for the space of aggregate changes in demand. And, because the set of all edge vectors is unimodular, this is an “integer basis”. So any integer change in aggregate supply can be presented as an integer combination of these basis vectors, thus assigning an integer change in bundle to each agent, although we have not yet demonstrated that each agent’s new bundle is in the convex hull of its demand set.

However, since we assumed the intersection was transverse at this price, the allocation of bundles to agents is unique. So the two allocations are the same. Thus both assign each individual agent an integer bundle in the convex hull of its demand, which the agent therefore demands, since its individual valuation is concave.

4.2.5 Sufficiency for the General Case

The full proof of sufficiency can be completed using the standard convex-geometric methods used thus far (see the Appendix of Baldwin and Klemperer, 2014). However, it is quickest to appeal to the tropical-geometric result that generically all THs intersect transversally:

**Proposition 4.6** (Maclagan and Sturmfels, 2015, Proposition 3.6.12). For any THs $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$, and generic $v \in \mathbb{R}^n$, the intersection of $\mathcal{T}_{u^1}$ and $\epsilon v + \mathcal{T}_{u^2}$ is transverse, for all sufficiently small $\epsilon > 0$.

So there always exist small perturbations of agents’ valuations that make their THs’ intersection transverse, so for which, by the previous argument, every bundle is demanded on aggregate at some price. But the aggregate value of any bundle that was not demanded before the perturbation must be a finite amount beneath the value that would be required for it to be demanded, which contradicts the bundle being demanded after an arbitrarily small perturbation of valuations.

Full details of the proof are in Appendix A.3.1.

4.3 Examples

4.3.1 Simple Illustration of Necessity of Theorem 4.2

We can illustrate our result on necessity of unimodularity for equilibrium by considering individual agents with the simple two-goods substitutes and complements valuations shown in Figs. 5a and 5b, respectively, and their aggregate valuation, shown in Fig. 5c. Note that both the individual valuations are concave. However, their aggregate valuation, given in Fig. 7a, is not, as can easily be seen by observing that $(U(1,0) + U(0,1) + U(2,1) + U(1,2))/4 > U(1,1)$. The failure of aggregate concavity is also clear in Fig. 7b, which shows the aggregate valuation together with the cell of its roof that corresponds to the price vector $(1,1)$.

It is apparent that all the bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are demanded at this price, but the bundle $(1,1)$ is not, and so also is never demanded at any price. So there is no equilibrium if the supply is $(1,1)$. 

24
Theorem 4.2 (and Corollary 4.3) warned of this possibility, since they imply equilib-
rium fails for some supplies and demands of any type containing this aggregate valuation.
The reason is that the minimal demand type containing both individual valuations con-
tains both (1, −1) (from the substitutes individual valuation) and (1, 1) (required for
the complements individual valuation), and no set of vectors containing both (1, −1)
and (1, 1) can be unimodular. Moreover, the discussion in Section 4.2.2 told us that
there would be an example of equilibrium failure of exactly this kind.

Working in quantity space, we can understand the failure of equilibrium by observing
that the matrix formed by (1, −1) and (1, 1) has determinant 2, so the demand type
is not unimodular. In particular, the area of the diamond formed by the edges in the
directions parallel to (1, −1) and (1, 1) in the SNP of the aggregate valuation has area
2—see Fig. 6c. So the bundle in the centre of the diamond, namely the quantity (1, 1) is
not a vertex of the aggregate SNP; it is correspondingly “hidden” at the intersection of
the diagonals at the price (1, 1) in the aggregate TH (in Fig. 5c), and it is indeed, in
this case, not demanded.

The equivalent price-space perspective is that we cannot move across TH facets
normal to (1, −1) and (1, 1) from, for example, the UDR in which in the quantity (0, 1)
demanded to any price at which the bundle (1, 1) is demanded, because aggregate
demand cannot change by (1, 0) (= (1, 1) − (0, 1)). The reason, as above, is that the
demand type is not unimodular and so, in particular, it is impossible to write (1, 0) as
a sum of integer multiples of (1, −1) and (1, 1) (but (1, 0) can be written as a sum of
real multiples of (1, −1) and (1, 1)).

4.3.2 Basis Changes

A benefit of our method of categorising valuations into ‘demand types’ is that it is
straightforward (see Appendix A.3.2) that:

(a) The aggregate valuation corresponding to Fig. 5c.

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(b) The aggregate valuation corresponding to Fig 5c, showing the cell of its roof that corresponds to the price vector (1/2, 1/2).

Figure 7: The aggregate valuation of Fig. 5c.
Proposition 4.7. “Having equilibrium for every set of agents with concave valuations and any supply bundle in the domain” is a property of a demand type that is preserved under unimodular basis changes.\textsuperscript{52}

Making such a basis change is equivalent to re-packaging the goods so that any integer bundle can still be obtained by buying and selling an integer selection of the new packages (and, conversely, any integer selection of the new packages will still be available as an integer combination of the original goods).

We illustrate below (Section 6.3) the usefulness of unimodular basis changes in finding new demand types for which equilibrium is guaranteed.

More generally a unimodular basis change simply distorts the TH by a linear transformation which leaves its important structure unaffected (see Proposition A.21 in the Appendix)\textsuperscript{53}. So other important properties of demand are also unaffected—see Baldwin and Klemperer (2014, in preparation-b).

4.3.3 Equilibrium with Complements

Mathematical results from Grishukhin et al. (2010) imply that every unimodular demand type is a unimodular basis change of a demand type that contains only vectors in $\pm\{0,1\}^n$ (and so contains only complements valuations). So from Proposition 4.7:

Proposition 4.8. Every demand type for which equilibrium is guaranteed (i.e., exists for every set of agents with concave valuations and any supply bundle in the domain) is a unimodular basis change of a demand type which contains only complements valuations and for which equilibrium is guaranteed.

Furthermore, the corresponding statement cannot be made about substitutes.\textsuperscript{54} This is in stark contrast to conventional wisdom about the “necessity” of substitutes for competitive equilibrium.\textsuperscript{55}

\textsuperscript{52}A unimodular matrix $G$ is an integer matrix with integer inverse. Premultiplying bundles of goods by $G$ is equivalent to premultiplying the prices at which bundles are demanded by $G^T$. This transforms the facet normals of the TH, and hence the vectors of any corresponding demand type, by $G^{-1}$. Details are in Appendix A.3.2.

\textsuperscript{53}We lay out the general behaviour in Appendix A.3.2. Analogous results about “basis changes” of valuations for divisible goods were developed by Gorman, 1976, pp. 219–220. Related results for specific cases of indivisible goods are in, e.g., Sun and Yang 2006, Sun and Yang 2008, and Hatfield et al., 2013.

\textsuperscript{54}We are very grateful to Tim O’Connor for proving this result.

\textsuperscript{55}For example, Gul and Stacchetti (1999, p. 96) state “in a sense, the GS [gross substitutes] condition is necessary to ensure existence of a Walrasian equilibrium”, and the specificity of their model in which such claims are valid often seems to be forgotten. And applying our results to matching (see Section 6.5, and Baldwin and Klemperer, 2014) shows that stable allocations arises for a broader class of preferences than many people assume from Hatfield and Milgrom’s (2005, p.915) statement “preferences that do not satisfy the substitutes condition cannot be guaranteed always to select a stable allocation”, though the Proposition (p.921) that their introductory remark loosely summarises is, of course, correct in its context.
4.3.4 Equilibrium with Strong Substitutes

It has been known for over 100 years (Poincaré, 1900) that the set of vectors which form our “strong substitutes” demand type (see Proposition 3.6) are a unimodular set. So Theorem 4.2 immediately implies the results of Danilov et al. (2001, 2003) and Milgrom and Strulovici (2009) that

**Proposition 4.9.** Equilibrium exists for every set of agents with strong substitutes valuations and any supply bundle in the domain.\(^{56}\)

Furthermore, the maximal set of vectors in a strong substitutes demand type is a maximal unimodular set. That is, adding any other vector, \(w\), that is not a strong substitutes vector, contradicts unimodularity: the determinant of \(w = e^i + e^j\) together with \(e^i - e^j\), and all \(e^k\) such that \(k \neq i, j\), has absolute value 2; and the determinant of any \(w\) with \(|w_j| > 1\) for some \(j\) with all \(e^i\) such that \(i \neq j\) is \(w_j\). So we also have Gul and Stacchetti’s (1999, Thm. 2), Hatfield and Milgrom’s (2005, Thm. 2), and Milgrom and Strulovici’s (2009, Thm. 16), result that:

**Proposition 4.10.** Given any one agent who does not have a strong substitutes valuation, there exist strong substitutes valuations for other agents such that equilibrium fails to exist for some supply in the domain.\(^ {57}\)

Danilov and Grishukhin (1999) provided a characterisation of all (what we call) unimodular demand types, including a list giving, up to unimodular basis change, all maximal such types up to dimension 6. From this list it is immediate that

**Proposition 4.11.** If \(n \leq 3\), equilibrium is guaranteed (i.e., exists for every set of agents with concave valuations and any supply in the domain bundle) for a demand type iff it is a unimodular basis change of strong substitutes, or a subset thereof.

With \(n > 3\), there exist demand types which are not a unimodular basis change of strong substitutes, or a subset thereof, for which equilibrium is guaranteed.

That is, while if there are at most three goods, all unimodular demand types are unimodular basis changes from strong substitutes, this is (as we already noted in Section 4.3.3) far from true more generally.

5 Existence of Equilibrium for Specific Valuations

Theorem 4.2 shows which demand types always have a competitive equilibrium, and for which demand types equilibrium fails for some valuations. For example, we saw equilibrium does not always exist for the demand type of Section 4.3.1, and indeed it did not exist for the valuations given there. But we will see below that if, for example, the “substitutes” agent of that example has a low enough valuation (e.g., the valuation \(u^*\) discussed in Section 3.3), then the demand type would be unchanged, but equilibrium would exist.

\(^{56}\)We show in Sections 6.1 and 6.3 that equilibrium existence results of Kelso and Crawford (1982), Hatfield and Milgrom (2005), Sun and Yang (2006, 2009), and Hatfield et al. (2013), are easy corollaries.

\(^{57}\)It also follows from Section 6.1 that Hatfield et al. (2013, Thm. 7) is an easy corollary.
So we now show that tropical intersection theory also provides results about *for which valuations* equilibrium exists, for demand types for which equilibrium does not always exist. As in our development of Theorem 4.2 (see Section 4.2), the key is that we can show that only certain isolated TH intersection points need be analysed. Moreover, tropical theory provides bounds on the number of such points and, remarkably, tells us that a simple count of them may suffice to demonstrate the existence or failure of equilibrium.

### 5.1 The Tropical Bézout-Kouchnirenko-Bernshtein theorem

The crucial point is that the celebrated Bézout theorem extends to tropical geometry: Bézout’s (1779) theorem tells us that in two dimensions the number of intersection points of two (ordinary) geometric curves equals the product of their degrees, except in degenerate cases when the curves have a component in common. In counting “intersection points”, we include those with complex coordinates and those at infinity (where, e.g., parallel lines “meet”) and in particular, we assign an appropriate “multiplicity” to each intersection point—for example, a tangency which is a “double” root (as, e.g., between a line and a parabola) has multiplicity 2 and so counts twice. So, for example, a quadratic (degree 2) and a line (degree 1) always intersect twice; a quadratic and a cubic (degree 3) intersect \(3 \times 2 = 6\) times. More recent (Bernshtein, 1975 and Kouchnirenko, 1976) versions of the theorem have extended it, including to curves in higher dimensions.

A deep insight of tropical geometry is that THs can be obtained as particular transformations of ‘ordinary’ geometric objects, and intersection properties are preserved under these transformations. Thus similar intersection theorems hold: once we have defined ‘multiplicity’ correctly, the TH provides exactly the right intersection counts. Furthermore, a TH is in real (not complex) space so we can “see them”.

Our contribution is to observe that a “too-high” multiplicity at a transverse TH intersection price corresponds to a failure of discrete-convexity of the demand set there, that is, a “hidden” bundle, and so a failure of equilibrium when the supply is that bundle. So, if the intersection is transverse, a sufficient number of intersection 0-cells guarantees equilibrium, and too few means that equilibrium may fail.

So, for example, Fig. 8 shows the TH of a “strong substitutes” valuation for up to 3 units in total, of 2 goods. All its facets have weight 1. Figs. 8b and 8c show the intersection of this TH with the THs of two different “strong substitutes” valuations for up to just 1 unit in total of the 2 goods. Notice that although the intersection prices are all in the same cell of the “1 unit” TH in Fig. 8b, and all in different cells of that TH in Fig. 8c, both TH intersections contain three points: it should be clear from these figures that any “1-unit strong substitutes” TH will intersect the first TH exactly three times if the intersection is transverse. (Recall from Propn. 3.6 that all “strong substitutes” facet normals for two goods are in \(\pm\{\{1,0\}, (0,1), (-1,1)\}\).) And this is precisely because the TH of any valuation for up to 3 units in total is the tropical transformation of an “ordinary” cubic, and the TH of any valuation for up to 1 unit in total is the tropical transformation of an “ordinary” line, and—as we already know from

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58THs are particular limits of logarithmic transformations of hypersurfaces (in complex projective space) in algebraic geometry. See, e.g., Maclagan and Sturmfels (2015) for a full discussion.
Figure 8: The THs of two generic valuations, one for up to 3 units, and one for a single unit, intersect exactly $3 \times 1 = 3$ times if equilibrium exists for any supply (as it does for strong substitutes valuations, illustrated).

Proposition 4.9—“strong substitutes” valuations always have equilibrium, so these THs must intersect exactly $3 \times 1 = 3$ times if their intersection is transverse.

By contrast, recall Figs. 5c and 5d giving the intersection between a single ‘complements’ agent with one of two possible ‘substitutes’ agents. In Fig. 5d, there are two intersections; in Fig. 5c there is only one. This corresponds precisely to the facts that competitive equilibrium exists for every supply in the former case, but—as we saw in Section 4.3.1—fails for some supply in the latter case.

5.2 Condition for Equilibrium to Exist for Every Supply, For a Given Set of Valuations

Bézout’s theorem was extended by Kouchnirenko and Bernshtein by use of “mixed volumes”:

**Definition 5.1.** The $n$-dimensional *mixed volume* of $n$ convex sets $X_1, \ldots, X_n \subset \mathbb{R}^n$ is

$$
MV_n(X_1, \ldots, X_n) = \sum_{k=1}^{n} (-1)^{n-k} \left[ \sum_{I \subset \{1, \ldots, n\}, |I| = k} \text{Vol}_n \left( \sum_{i \in I} X_i \right) \right].
$$

Write $MV_n(X, Y, (k, n-k))$ for the mixed volume of $k$ copies of $X$ and $n-k$ copies of $Y$, for any $0 \leq k \leq n$.

Here, $\text{Vol}_n(X_i)$ is the $n$-dimensional volume of $X_i$ (so $\text{Vol}_n(X_i) = 0$ if $\text{dim}(X_i) < n$). So, in two dimensions, $MV_2(X, Y) = \text{Vol}_2(X+Y) - \text{Vol}_2(X) - \text{Vol}_2(Y)$ (in which $\text{Vol}_2(.)$ is, of course, just the two-dimensional area), and in three dimensions, $MV_3(X, Y, Z) = \text{Vol}_3(X+Y+Z) - \text{Vol}_3(X+Y) - \text{Vol}_3(Y+Z) - \text{Vol}_3(Z+X) + \text{Vol}_3(X) + \text{Vol}_3(Y) + \text{Vol}_3(Z)$, etc. Thus the “mixed volume” is a linear combination of ordinary volumes. An important special case is that it can be shown that $MV_n(X, \ldots, X) = n! \text{Vol}_n(X)$. See Appendix A.4.3 for further discussion.

We also generalise the concept of “weight”, that we previously defined for TH facets, so that it applies to any TH cell. We do this by letting the *weight* of a $(n-k)$-cell of a TH be the $(k$-dimensional) volume of the corresponding SNP cell divided by the $(k$-dimensional) volume of its fundamental simplex.\(^{59}\) We will give examples of use of mixed volumes, and weights, below.

\(^{59}\)A ‘simplex’ on $r$ vectors is the convex hull of those vectors together with $\mathbf{0}$. For the fundamental
Finally, we define the “naïve weighting” of any 0-cell at which two THs, \( T_{u^1} \) and \( T_{u^2} \), intersect transversally, as \( w_1 w_2 \) in which \( w_i \) is the weight of the minimal cell of \( T_{u^i} \) that contains the 0-cell.\(^{60}\)

As prefigured in the previous subsection (5.1) the tropical Bézout-Kouchnirenko-Bernshtein theorem, developed by Bertrand and Bihan (2007, 2013), now leads to a straightforward way to determine whether equilibrium exists for valuations whose THs’ intersection is transverse. We first give the important theorem, and then explain it in more detail in Section 5.3 (full details are in Appendix A.4).

The key geometric result requires (only) that for some fixed \( k \), the intersection of two THs is transverse at all 0-cells which are an intersection of a \( k \)-cell of the first TH, and an \((n-k)\)-cell of the second:

**Lemma 5.2.** Let \( u_1 \) and \( u_2 \) be any concave valuations on domains whose convex hulls are \( \tilde{A}_1 \) and \( \tilde{A}_2 \), and such that the domain of aggregate demand has dimension \( n \), and such that for some \( k \in \{1, \ldots, n-1\} \), any intersection between a \( k \)-cell of \( T_{u^2} \) and a \((n-k)\)-cell of \( T_{u^2} \) is transverse.

The naïvely-weighted count of 0-cells at such cell intersections is bounded above by \( MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k)) \). If this count equals this upper bound, then equilibrium exists for \( u_1 \) and \( u_2 \) and any supply in the convex hull of demand at any such price. If the bound is not met with equality and \( n \leq 3 \), then equilibrium fails for some supply in the convex hull of demand at some such price.

It follows immediately from summing over \( k \) (and since a TH has no \( n \)-cells) that:

**Theorem 5.3.** Let \( u_1 \) and \( u_2 \) be any concave valuations on domains whose convex hulls are \( \tilde{A}_1 \) and \( \tilde{A}_2 \), and such that the domain of aggregate demand has dimension \( n \), and whose TH intersection is transverse.

The naïvely-weighted count of 0-cells in their TH intersection is bounded above by \( \sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k)) \). If this count equals this upper bound, then equilibrium exists for \( u_1 \) and \( u_2 \) and any supply in the domain. If the bound is not met with equality and \( n \leq 3 \), then equilibrium fails for some supply in the domain.

To test whether equilibrium is guaranteed to exist for \( m > 2 \) valuations, we can sequentially check whether it always exists for the \((l+1)\)th valuation and the aggregate of the first \( l \) valuations, for \( l = 1, \ldots, m-1 \). Of course, if the aggregate domain is of dimension less than \( n \), we can just make a basis change to reduce the dimension of the goods-space so that the aggregate domain is of full dimension, and Theorem 5.3 (and Lemma 5.2) apply.

For example, for \( n = 2 \), the cell weights are just the facet weights. If, also, \( \tilde{A}_i \) contains all bundles of up to a total of \( d_i \) units, \( i = 1, 2 \), it is straightforward that \( \text{Vol}_2(\tilde{A}_i) = (d_i)^2/2 \), that \( \tilde{A}_1 + \tilde{A}_2 \) is the convex hull of all bundles of up to a total of \( d_1 + d_2 \) units so \( \text{Vol}_2(\tilde{A}_1 + \tilde{A}_2) = (d_1 + d_2)^2/2 \), and so \( \sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k, k)) = \) simplex, the relevant vectors are any integer basis of the minimal linear space parallel to the SNP cell. Its volume is \( 1/(k!) \) of that of the fundamental parallelepiped, namely the parallelepiped whose edges are this basis. See Definition A.31.

\(^{60}\)Note this weight is not the “multiplicity” of the tropical Bézout-Kouchnirenko-Bernshtein theorem; indeed it is the distinction between them that yields our results.
Suppose two valuations for up to a total of \(d_1\) units, and \(d_2\) units, respectively, of two goods have a transverse TH intersection (as is generic). Then the number of points in their intersection, each weighted by the product of the weights of the facets that intersect at that point, equals \(d_1 d_2\) if equilibrium exists for these valuations for every supply, and is lower than \(d_1 d_2\) otherwise. The simple example illustrated in Figs. 8b-c in the previous subsection (5.1) is a special case.

5.3 Explanation and Illustration of Theorem 5.3

Theorem 5.3 is an application of the tropical Bézout-Kouchnirenko-Bernshtein theorem, which tells us that if two THs’ intersection is transverse, then the “multiplicity”-weighted count of 0-cells in the intersection equals the relevant mixed volume. Our contribution is demonstrating that the multiplicity-weighting is equal to the “naïve” weighting precisely when equilibrium is guaranteed (or, when \(n \leq 3\), precisely when it exists).

It is easiest to illustrate it in the two good case. Then a transverse intersection of two THs consist only of 0-cells, the “multiplicity” of such a cell is just the area of the corresponding aggregate SNP cell, and these SNP cells are all parallelograms with integer area. Furthermore, the minimal (indeed only) cells of the individual THs that contain these intersection points are the corresponding facets, so the relevant cell weights are just the facet weights.

Begin with the case in which all the facet weights are 1, so the naïvely-weighted count of Theorem 5.3 is just the unweighted count. It follows from the tropical Bézout-Kouchnirenko-Bernshtein theorem that if the equality of Theorem 5.3 holds, the areas of the SNPs at the intersection points must all be 1 (since they must all be positive integers). So, as argued in Section 4.2 above, the aggregate demands are discrete-convex at all these points so, by Lemma 4.4, equilibrium exists for every supply.

But if the equality of Theorem 5.3 fails, then some multiplicity, and hence the area of some SNP at an intersection price, must exceed 1. So there is a “hidden bundle” at this price, and equilibrium must fail for this supply bundle. The reason is exactly as explained at Section 4.2.2: both agents have just two bundles in their demand set at this price, these can create only the \(2 \times 2 = 4\) possible aggregate demands that correspond to the four vertices of the aggregate SNP cell, and so the “hidden bundle” cannot be an aggregate demand.

For example, consider the simple two-good substitutes and complements valuations, \(u^s\) and \(u^c\), whose individual THs and aggregate TH are shown in Figs. 5a-5c, and whose individual SNPs and aggregate SNP are shown in Figs. 6a-6c. The domain of each individual valuation is \(\{0, 1\}^2\), so the domain of the aggregate is \(\{0, 2\}^2\), and the relevant mixed volume is therefore \(4 - 1 - 1 = 2\). The individual THs intersect only at the price \((\frac{1}{2}, \frac{1}{2})\), and this is transverse (so Theorem 5.3 applies). So, since the two facets containing it both have weight 1, the naïvely-weighted count is just 1, and equilibrium therefore fails for some supply. Indeed, since the relevant mixed volume is 2, we know that the area of the (aggregate) SNP cell at the single intersection point must be 2, as we can see in Fig. 5c; the relevant cell is the central diamond, and contains the “hidden”

\[ MV_2(\tilde{A}_1, \tilde{A}_2) = d_1 d_2. \]

More generally, if \(\tilde{A}_i\) contains all bundles of \(n > 2\) goods up to a total of \(d_i\) units, \(i = 1, 2\), then

\[ \sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k)) = \sum_{k=1}^{n-1} d_1^{n-k} d_2^k. \]
bundle \((1, 1)\) at which equilibrium fails, as we saw in Section 4.3.1.

On the other hand, \(u^*\) (see Section 3.3) is a valuation of the same demand type (indeed same combinatorial type), and on the same domain as \(u^\ast\), so the mixed volume relevant to testing for equilibrium of \(u^*\) and \(u^\ast\) is still 2, but the individual THs now intersect at two prices, \((\frac{1}{4}, \frac{3}{4})\) and \((\frac{3}{4}, \frac{1}{4})\)—see Fig. 5d. Both these intersection points are transverse, so Theorem 5.3 still applies, and the total na"ıvely-weighted count is now 2. Equilibrium therefore exists for every supply, as can be seen in Fig. 6d: the two SNP cells corresponding to the intersection points are the two parallelograms which both have area 1, and every supply bundle is a vertex of the SNP, and so is demanded for some price.

It is not hard to generalise to the case in which the intersecting facets’ weights, \(w_1\) and \(w_2\), may exceed 1. Because we assumed that the individual agents have concave valuations, each agent, \(j\), is indifferent, at prices on its facet, among the \((w_j + 1)\) bundles on the corresponding edge in its individual SNP.\(^{62}\) Agent, \(j\)’s SNP edge is, of course, of “length” \(w_j\) (that is, it is \(w_j\) times its primitive integer vector), and the SNP cell of the aggregate valuation is now a “large” parallelogram, which can be divided by a grid into \(w_1w_2\) copies of a small parallelogram whose edges are the minimal (i.e., primitive) integer vectors in the same directions.

It is clear that all the bundles on the vertices of this grid are demanded at the intersection price; they correspond to the different ways in which we can give either agent one of the bundles between which it is indifferent. (Again, these bundles correspond to “grey” bundles, not “black” ones, in our discussion in Section 2.4.) Furthermore, there are no other bundles in the “large” parallelogram if and only if each of the (identical) small parallelograms has area 1. So there is no problem bundle at the intersection price if and only if the area of the large parallelogram (that is, its multiplicity) equals \(w_1w_2\) (that is, the naïve weighting of the corresponding 0-cell). It follows again, therefore, that equilibrium holds for every supply in the domain of the aggregate valuation if the naïvely-weighted count of all the intersection 0-cells equals the multiplicity-weighted count, and the tropical Bézout-Kouchnirenko-Bernshtein theorem tells us the latter equals the relevant mixed volume.

For more than two goods, the logic is similar in spirit, but more complicated in detail:

First, the definition of cell weight is more intricate, and so is that of “multiplicity”.

Next, the intersection does not consist only of 0-cells, but (in the transverse case) will be a ‘rational polyhedral complex’ (that is, have a similar structure to a lower-dimensional TH) of dimension \((n - 2)\). So we first simplify the situation. To do this, we recall (Lemma 4.4) that we are focussing on intersection prices because, if equilibrium fails for any supply, there must be a failure of discrete-convexity of the aggregate demand at some price in the intersection. Any such price lies in some \(k\)-cell of the aggregate TH. It follows easily from the duality construction of Section 2.4 (see Corollary A.11) that because we assume the domain of the aggregate valuation is \(n\)-dimensional, this \(k\)-cell has some 0-cell(s) in its boundary (which hence also lie in the individual THs’ intersection). And it also follows that the bundle demonstrating failure of discrete-convexity at the original price also demonstrates failure of discrete-convexity at the

\(^{62}\)The reason is the same as for the “grey” bundles, not “black” ones, in our discussion in Section 2.4: \(j\)’s valuation is linear along the SNP edge.
0-cell price. So we can restrict attention to just the 0-cells of the intersection, that is, a finite number of points.

Finally, a transverse intersection 0-cell in more than two dimensions need not correspond to a parallelepiped in the SNP. However, we can draw a parallelepiped which ‘almost’ gives the right information. One vertex is at an aggregate demand at this 0-cell price, and the edges comprise the bases for possible changes in demand for each individual at this price. Some subset of this parallelepiped is contained in the aggregate SNP cell corresponding to the 0-cell of the TH at the intersection.

The parallelepiped contains additional bundle(s) to those that are either on its vertices or on the vertices of a grid of “small” parallelepipeds (in the case that any of its edges’ weights exceed 1) if its volume exceeds the product of its edges’ weights. One can show that, if the parallelepiped does not contain such a potentially-problematic bundle, then there are no problems in the SNP cell itself. However, if the parallelepiped does contain such a bundle, these bundle(s) may or may not lie in the SNP cell itself.

In three dimensions this creates no ambiguity: the TH cell at the intersection is at least half a parallelepiped (and its vertices are vertices of the parallelepiped). By symmetry, if there is any additional bundle in a parallelepiped then there is one in both halves. But, in four or more dimensions, the equality of the Theorem 5.3’s condition is sufficient, but no longer necessary, for equilibrium, as we illustrate in Example A.29 in the Appendix. However, we can then use Lemma 5.2 for each individual \( k \) separately, to narrow down where, if anywhere, equilibrium might fail; we make use of this in the next subsection, where we create a general recipe for checking the existence of equilibrium.

### 5.4 Checking Equilibrium in the General Case

It might be conjectured that we could apply Theorem 5.3 generally, analogously to our analysis in Section 4.2.5 that used the fact that if equilibrium fails at an intersection price, then it also fails after a small perturbation in valuations that makes the intersection transverse. However, it is not true that if equilibrium exists, it also exists after a small generic perturbation: there are ‘fragile’ equilibria which only arise at a non-transverse intersection (see Appendix Example A.36). To determine whether equilibrium exists for such cases, we first need a little more tropical-intersection theory:

**Definition 5.4.** The stable intersection, \( \mathcal{T}_{u^1} \cap_{st} \mathcal{T}_{u^2} \), of THs \( \mathcal{T}_{u^1} \) and \( \mathcal{T}_{u^2} \) consists of all cell intersections \( C^1 \cap C^2 \) where \( C^i \) is a cell of \( \mathcal{T}_{u^i} \) and \( \dim(C^1 + C^2) = n \).

\[ ^{63} \text{We conjecture that there are other “counting” methods of determining whether equilibrium exists when the intersection is transverse. For example, we could perturb each agent’s original valuation by a small amount to make it “strictly concave”, that is, so that all the cells of the individual TH where “hidden” bundles just touch the roof are “opened up” (so, for example, all the facet weights of the individual THs are then 1, and all bundles that correspond to “grey” ones in our earlier discussion of Section 2.4 turn “white”). And we could make the perturbation small enough that no new facet that has been created (or moved) in either individual TH is far enough from the location of the original facet from which it was derived to otherwise disturb the structure of the aggregate TH. We could then check whether the number of UDRs in the new aggregate TH equals the number of bundles in the aggregate domain. However, this would both be cumbersome, and also require detailed knowledge of the aggregate TH to ensure the perturbation was small enough. Moreover, there is no obvious way to extend such an approach to analyse the general case which we now discuss.} \]
Recall that two THs intersect transversally at \( p \) if \( C^j \) is the minimal cell of \( T_u \), containing price \( p \) and \( \dim(C^1 + C^2) = n \). So the stable intersection of two THs contains all cells at which they intersect transversally, and may contain some or all of any additional cells where they intersect.

For example, in Fig. 5c, if the (downward-sloping) “complements” TH were translated by a parallel shift to pass through \( (1,1) \)–the vertex where three line-segments of the “substitutes” TH meet–then the THs would not intersect transversally anywhere (the intersection at \( (1,1) \) is non-transverse since the substitutes TH has a 0-cell there). However, the point \( (1,1) \) would be the stable intersection, since it is the intersection of the “complements” line (1-cell) with any one of the three “substitutes” line-segments (1-cells). Two identical substitutes THs, both of the kind shown in Fig. 5a, also have no transverse intersection, while the 0-cells \( (0,0) \) and \( (1,1) \) would then form the stable intersection. Note that the coincident line segments of the intersection are not in the stable intersection. More generally, a stable intersection of two THs in \( n \) dimensions has the same ‘cell structure’ as a TH (that is, it is a rational polyhedral complex) and is of dimension \( n - 2 \).

An alternative way to define the “stable intersection” both gives additional insight into the relationship with transverse intersections, and yields an important result. The “stable intersection” is the limit-set of any series of “perturbed” TH intersections that can be created by fixing one of the THs, and imposing on the other a series of arbitrarily small (and decreasing to zero) translations such that in every case the intersection between the THs is transverse. It is a standard result of tropical geometry that this limit is well-defined.

Now recall that we showed (in our proof of Theorem 4.2) that equilibrium can fail at an intersection only if it also fails for arbitrarily close valuations that intersect transversally. So if equilibrium fails at an intersection, we can take a series of infinitesimally small and decreasing perturbations that yield transverse intersections at which it also fails. It follows that equilibrium must also fail in the limit of these transverse intersection prices, that is, at the stable intersection. So any failure of equilibrium must show up at prices in the stable intersection (since, from Lemma 4.4, we only need to check intersections of THs). That is:

**Theorem 5.5.** If the domain of the aggregate valuation of two concave valuations has dimension \( n \), then equilibrium exists for every supply bundle in the domain iff the aggregate demand set is discrete-convex at every 0-cell of the stable intersection of agents’ THs.

In particular, if equilibrium fails for a supply \( x \), then this supply must exhibit failure of discrete-convexity (that is, \( x \in \text{Conv} D_U(p) \), but \( x \notin D_U(p) \)) at some price \( p \) in such a 0-cell. As usual, if the aggregate domain is of dimension less than \( n \), we can just make a basis change to reduce the dimension of the goods-space so that the aggregate domain is of full dimension, and Theorem 5.5 then applies.

So even if the intersection of two THs contains a continuum of points, we only need

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\(^{64}\)See Maclagan and Sturmfels, 2015, Thm. 3.6.10. If \( n = 1 \) then the stable intersection is empty.

\(^{65}\)Maclagan and Sturmfels’ 2015 Prop. 3.6.12, stated as our Appendix Proposition A.37. Note this definition gives us another easy way to see, in the examples just above, that the 0-cell at price \((1,1)\) is in the stable intersection, but is not transverse.
to check a finite number of points to find out whether equilibrium always exists.\footnote{But if equilibrium does fail at such a point, it might then also fail at a continuum of prices.} And, as before, to test whether competitive equilibrium always exists for \( m > 2 \) valuations, we can sequentially check, for \( l = 1, \ldots, m-1 \), whether equilibrium always exists for the aggregate of the \((l+1)^{th}\) valuation and the aggregate valuation of the first \( l \) valuations.

Furthermore, the Tropical Bézout theorem now gives us a bound on the number of the points that we have to check (see Appendix)

\[ \text{Theorem 5.6 (cf. Bertrand and Bihan, 2007, 2013). The number of 0-cells in the stable intersection of THs } T_{u1} \text{ and } T_{u2} \text{ of valuations on domains whose convex hulls are } \tilde{A}_1 \text{ and } \tilde{A}_2 \text{, respectively, which are inside a } k \text{-cell of } T_{u1} \text{ and a } (n-k) \text{-cell of } T_{u2} \text{, is bounded above by } MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k)). \text{ The total number of 0-cells in the stable intersection of the THs is bounded above by } \sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k)). \]

The second part of this result is obvious from the fact that any 0-cell in the stable intersection of two THs is contained in a \( k \)-cell of one TH and an \((n-k)\)-cell of the other TH for some \( 1 \leq k \leq n - 1 \) (see Definition 5.4; a TH has no \( n \)-cells, of course), although \( k \) need not be uniquely defined.

Observe that the two-dimensional case \( (n = 2) \) is particularly straightforward; the bound on the total number of 0-cells is then just \( MV_2(\tilde{A}_1, \tilde{A}_2) \).

In sum, therefore:

\[ \text{Corollary 5.7. For any concave valuations, we can check whether or not equilibrium exists at every supply in the domain by checking only a finite set of prices, the number of which we have bounded.} \]

Moreover, combining Theorems 5.3, 5.5 and 5.6 and Lemma 5.2 yields a recipe to test whether equilibrium exists for every supply in the domain, for the aggregate of any two concave valuations:\footnote{Of course, if 5(i) holds, then so does 5(ii). Also, if \( n \leq 3 \), then if 5(i) fails for any \( k \) (for which every 0-cell in a \( k \)-cell of \( T_{u1} \) and a \((n-k)\)-cell of \( T_{u2} \) is a transverse intersection point), then so will 5(ii), so there is no need to proceed further.}

\[ \text{Recipe 5.8. (1) If the domain of the aggregate valuation is in less than full dimension, make a basis change so that it is in full dimension.} \]

\[ (2) \text{ If the intersection is not transverse, go to (5).} \]

\[ (3) \text{ If the naively-weighted count of 0-cells in the intersection equals } \sum_{k=1}^{n-1} MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k)), \text{ equilibrium exists for all supplies.} \]

\[ (4) \text{ If not, and } n \leq 3, \text{ equilibrium fails for some supply.} \]

\[ (5) \text{ Equilibrium exists for all supplies iff for every } k \]

\[ (i) \text{ every 0-cell in a } k \text{-cell of } T_{u1} \text{ and a } (n-k) \text{-cell of } T_{u2} \text{ is a transverse intersection point, and the naively-weighted count of these 0-cells equals } MV_n(\tilde{A}_1, \tilde{A}_2, (n-k,k)) \]

or \( (ii) \text{ for every 0-cell in a } k \text{-cell of } T_{u1} \text{ and a } (n-k) \text{-cell of } T_{u2}, \text{ the aggregate demand set (found from the two agents’ individual demand sets) can be seen directly to be discrete-convex.} \]
6 Applications

6.1 Interpreting Classic Models in a Unified Framework

Our model encompasses some classic studies as special cases, so clarifies connections between them. It also facilitates our understanding of these papers. In particular, it makes many of their equilibrium-existence results straightforward. And Baldwin and Klemperer (2014, in preparation-b) use our framework to study the implications of their assumptions on preferences.

Kelso and Crawford’s (1982) seminal analysis of $n_1$ firms, each of which is interested in hiring some of $n_2$ workers, can be understood as a model with $n_1 n_2$ distinct “goods”, each of which is the “transfer of labour” by a specified worker (a “seller”) to a specified firm (a “buyer”); the “price” of a good is the salary to be paid. So the full set of bundles we consider is $\{-1, 0, 1\}^n$, in which $n = n_1 n_2$, but each agent’s valuation is defined only on the subset of this domain that is relevant to it.

Specifically, each worker has preferences only over a subset of the domain of the form $\{-1, 0\}^{n_1}$ (that is, it has preferences only over the $n_1$ goods that correspond to its own labour), and only over the subset of these vectors that have at most one non-zero entry (it can work for at most one firm). Obviously, their only possible SNP edges are vectors of the strong substitutes demand type (at most one +1 entry, at most one -1 entry, and no other non-zero entries, see Proposition 3.6). Furthermore, each firm only has preferences over a subset of the aggregate domain of the form $\{0, 1\}^{n_2}$ (that is, it has preferences only over the $n_2$ goods that correspond to workers it employs to work for itself). Moreover, Kelso and Crawford assume firms have ordinary substitutes preferences over workers, and the only edges of firms’ SNPs that are vectors of the ordinary substitutes demand type are clearly also vectors of the strong substitutes demand type.\(^{68}\)

It is perhaps less obvious that Hatfield et al.’s (2013) model of networks of trading agents, each of whom can both buy and sell, both fits into our framework, and is also closely related to Kelso and Crawford’s model. To show this is so, we (again) treat each transfer of a product from a specified seller to a specified buyer as a separate good, so each agent again has preferences over a subset of $\{-1, 0, 1\}^n$, where $n$ is now the number of ‘separate goods’.

Since Hatfield et al. restrict each agent to be either a seller or a buyer (or neither) on any one good, an agent $i$ which is the specified seller in $n_1^i$ potential trades and is the specified buyer in $n_2^i$ potential trades simply has preferences over a subset of the domain which, after an appropriate re-ordering of the goods for that agent, is of the form $\{-1, 0\}^{n_1^i} \times \{0, 1\}^{n_2^i}$. (As in Hatfield et al., we can restrict an agent’s domain of preferences further so that, e.g., it cannot sell good 1 unless it also buys one of goods 2 or 3.) Furthermore, although Hatfield et al. describe goods to be sold as complements of goods to be bought, this is because they measure both buying and selling as non-negative quantities. So, since in our framework selling is just “negative buying”, the “complementarities” disappear and it is clear that the condition they impose is exactly

\(^{68}\)Kelso and Crawford actually make a more restrictive assumption than this for their substitute preferences, but in fact the characterisation follows from the weaker assumption mentioned here. See Danilov et al. (2003), Baldwin and Klemperer (2014), and Baldwin, Klemperer and Milgrom (in preparation).
ordinary substitutes. Just as for Kelso and Crawford’s model, the only SNP edges of such a domain that are vectors of the ordinary substitutes demand type are also vectors of the strong substitutes demand type.

Trivially, any valuation over any subset of \(-1, 0\)^n_1 or \(0, 1\)^n_2 or \(-1, 0\)^n_1 × \(0, 1\)^n_2 is concave so, in both Kelso and Crawford’s and Hatfield et al.’s models, the existence of equilibrium follows immediately from Proposition 4.9.

Reformulating models in our framework also shows clearly how we can generalise them. It is immediate, for example, that as long as we retain concavity and the strong substitutes demand type, we can remove Hatfield et al.’s restriction that an agent cannot be both a buyer and a seller on any one good (by simply extending their domain to be any subset of \((-1, 0, 1)^n\)) and can also permit their agents to trade multiple units of the same products (by enlarging the domain to any subset of \(\mathbb{Z}^n\)).

Other models that fit into our framework are Bikhchandani and Mamer (1997) (this is just the restriction of our model to \(A = \{0, 1\}^n\), and Hatfield and Milgrom’s (2005) famous model of ‘contracts’ (since this can be embedded in Kelso and Crawford’s model—see Echenique, 2012).

### 6.2 Analysing when Equilibrium Exists

As an example, consider “complements” consumers, each of whom is only interested in a single, specific, pair of goods. If there is a cycle in these pairs, we can number both goods and kinds of consumers \(1, \ldots, n\), such that every consumer of kind \(i < n\) demands goods \(i\) and \(i + 1\), which it sees as perfect complements, and consumers of kind \(n\) demand goods \(n\) and 1. It is easy to check that:

\[
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{array}
\]

\[
\det = \left\{ \begin{array}{ll}
0 & \text{if } n \text{ is even} \\
2 & \text{if } n \text{ is odd.}
\end{array} \right.
\]

So if \(n\) is odd, Corollary 4.3 tells us equilibrium does not always exist. Furthermore, our proof of Theorem 4.2 shows that we can find an example of equilibrium failure by simply selecting a single consumer of each type, each of which values its desired pair at \(v\), so that they are all indifferent between purchase and no purchase (and hence their facets all intersect) if every good’s price is \(v/2\). But the tropical-Bézout methods of Section 5 can determine, for any given set of agents’ valuations, whether equilibrium

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69Their ‘choice language’ definition differs superficially from Definition 3.2, but Hatfield et al., 2011, Thm A.1 confirms the equivalence.

70So equilibrium fails if aggregate demand is exactly 1 unit of each good (the “middle of the parallelepiped”) since the minimum and maximum aggregate demands are zero, and 2 units of each good, respectively, at this price. (It is easy to check failure of equilibrium for \(x_1 = 1\), for all \(i\), directly: at least one good, w.l.o.g. good 1, would not be part of a pair. So \(p_1 = 0\). Therefore \(p_2 \geq v\), since otherwise consumer 1 would demand both goods 1 and 2. So \(p_2 = v\), and therefore \(p_3 = 0\), since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore \(p_4 \geq v\), etc., so \(p_j = 0\) if \(j\) is odd. But consumer \(n\) then wishes to buy goods \(n\) and 1, which is a contradiction.)
exists for every supply in the domain.

If \( n \) is even, the columns of this matrix are not linearly independent. However, if we exclude the \( i \)th column, for any \( i \), the remaining \( n - 1 \) rows are then linearly independent, and can trivially be extended to \( n \) linearly independent vectors with determinant 1 by adding the column \( e^i \). So using Theorem 4.2, equilibrium always exists if \( n \) is even, since the valuations are, trivially, concave.\(^{71,72}\)

### 6.3 Basis Changes to find new Demand Types that always have Equilibrium

We can use the fact (Section 4.3.2) that equilibrium existence, and other properties,\(^{73}\) are unaffected by basis changes, together with knowledge of these properties for any one demand type, to obtain useful results about other demand types.\(^{74}\)

For example, a number of transformations of strong substitutes are of interest:

**Interval Package Valuations**\(^{75}\) Premultiplying the vectors of the strong substitutes demand type, that is, \( e^i \) and \( (e^i - e^j) \), by the upper triangular matrix of 1s (of the appropriate dimension) yields the vectors \( \sum_{k=1}^i e^k \) and \( \sum_{k=j+1}^i e^k \) for \( i > j \), respectively (and their negations). This is the “interval package valuations” demand type for goods which have a natural fixed order, and for which any contiguous collection of goods may be considered as complements by any agent. For example, valuations for bands of radio spectrum, or for ‘lots’ of sea bed to be developed for offshore wind (see Ausubel and Cramton, 2011) may be of this form.

So this is an important purely-complements demand type for which equilibrium always exists when valuations are concave.

**Generalised Gross Substitutes and Complements** Premultiplying the vectors of the maximal strong substitutes demand type by a matrix formed of \( \{e^i \mid i \leq k\} \cup \{-e^i \mid i > k\} \), for some \( k \), yields the demand type whose valuations are those satisfying Sun and Yang’s (2006, see also 2009) definition of “gross substitutes and complements” extended to permit multiple units of goods, and sellers as well as buyers.\(^{76}\) These are valuations such that goods can be separated into two groups, with goods within the same group being strong substitutes, and each good also may exhibit 1:1 complementarities with any good in the other group. As above, it is immediate that equilibrium always exists

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\(^{71}\)For example, the aggregate demand of 1 unit of each good is supported by price \( v/2 \) for every good, when there is exactly one consumer of each kind, each of which values its preferred pair at \( v \).

\(^{72}\)Sun and Yang (2011) and Teytelboym (2014) have independently used alternative methods to show these results for a version of this model; the even \( n \) case is also a special case of the “generalised gross substitutes and complements” demand type that we discuss in Section 6.3.

\(^{73}\)See Baldwin and Klemperer (2014, especially Section 5) and Baldwin and Klemperer (in prep-n-c).

\(^{74}\)We implicitly gave an example above, when we observed that Hatfield et al’s (2013) model of complements could be understood as being restricted to strong substitutes by relabelling the good “purchase of a unit” as \(-1\) units of a good “sale of a unit”.

\(^{75}\)These valuations were introduced by Danilov et al. (2008, 2013).

\(^{76}\)The demand type’s vectors are \( \{e^i, e^i', e^i - e^j, e^i + e^j, e^i - e^j' \mid i, i' \in \{1, \ldots, k\}, j, j' \in \{k + 1, \ldots, n\}\} \).
6.4 Other new Demand Types

Recall from Proposition 4.11 that there are also many unimodular demand types that are not a unimodular basis change of strong substitutes. One example, containing only complements demands, that has not previously been studied, is the demand type, defined by the columns of the matrix $D$ (and their negations).

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

This might, for example, model the demand by firms for “bundles of” four kinds of workers—three sorts of specialist (the first three goods) and a supervisor (the fourth good). The first three columns of $D$ show that any one of the three kinds of specialist has value on his own; a supervisor on her own is worthless; but the middle three columns of $D$ show that a supervisor increases the value of any specialist (that is, there are pairwise complementarities between any one of the first three ‘goods’ together with the fourth); and the last three columns of $D$ show that there are also complementarities between any pair of different specialists if (but only if) a supervisor is also present.

6.5 Matching

The example in the previous subsection can be interpreted as a multi-player matching problem in which the columns of $D$ are the coalitions of workers that create value. Baldwin and Klemperer (2014, in preparation-a) show that, assuming perfectly transferable utility, a stable matching in which no subset of workers can gain from re-matching (that is, an allocation in the core of the game among the workers) corresponds exactly to a competitive-equilibrium allocation of workers in our model (in which every worker receives its competitive wage, and no further gains from trade are possible). So, since

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\[77\] Shioura and Yang (2013) have independently made the same extension of gross substitutes and complements, and shown that equilibrium always exists for it.

\[78\] because $e^4 \notin D$; perhaps each firm’s owner is a supervisor herself, and an additional supervisor without any workers would merely “spoil the broth”.

\[79\] There is an infinite family of related unimodular demand types in higher than four dimensions. Note that there are, of course, many basis changes of $D$ (and of any unimodular demand type) that include all the coordinate vectors (Appendix A.6 gives an example). However, $D$ is not a unimodular basis change of any substitutes demand type (we thank Tim O’Connor for showing this; for a simple demonstration that it is not a unimodular basis change of a strong substitutes demand type, see Appendix A.6).

Another intriguing example of a unimodular demand type that is not a member of this family, but which is also not a unimodular basis change of strong substitutes, contains all vectors in $\mathbb{Z}^6$ with three +1 entries and three -1 entries, thus incorporating (all) valuations for six distinct goods in which all changes in an agent’s demand involve swapping three of the goods for the other three goods as prices move between UDRs. Danilov and Grishukhin’s (1999) characterisation of maximal unimodular sets of vectors provides many more examples (including a basis change of $D$, but not a pure-complements one with this interpretation).
the demand type is unimodular, it describes a class of multi-player matching problems for which a stable match always exists.

More generally, Baldwin and Klemperer (2014, in preparation-a) show any matching problem with perfectly transferable utility corresponds to a demand type containing only vectors in $\pm\{0,1\}^n$. If, as in the “workers” example above, the demand type is unimodular a stable match always exists. If it is not unimodular, the tropical-Bézout methods of Section 5 tell us for what coalitions’ valuations there are stable matchings.\(^{80}\)

\section{Understanding Individual Demand}

Our techniques are powerful tools for understanding individual demand. In particular, Mikhalkin’s important observation (our Theorem 2.3) tells us that any balanced rational polyhedral complex is the TH of some quasilinear valuation and conversely. This allows us to explore properties of valuations by drawing and analysing appropriate geometric diagrams without needing to undertake the typically much more challenging task of constructing valuations that generate these diagrams.

Baldwin and Klemperer (2014, in preparation-b) further explores the comparative statics of individual demand, in order to better understand demand changes at non-UDR prices. Unimodularity turns out to have important implications for the structure of individual demand, as well as (as we saw in our discussion of the existence of equilibrium) for aggregate demand. This work also leads to a generalisation of Gul and Stacchetti’s (1999) “Single Improvement Property”.

Related work (joint with Paul Milgrom) uses our framework to help understand implications of different notions of substitutability for indivisible goods that have been suggested in the literature.\(^{81}\)

\section{Auction Design}

Practical auctions need to restrict the kinds of bids that can be made, thus restricting the preferences that bidders can express. Restricting to a demand type is often natural, since the economic context often suggests appropriate trade-offs between goods. For example, the Bank of England expected bidders to have £1:£1 trade-offs between any pair of the several different “kinds” of money it loaned in the financial crisis.\(^{82}\) So it chose auction rules that made it easy for bidders to communicate strong substitutes preferences, and was unconcerned about ruling out the expression of other preferences.\(^{83}\)

Knowing that the bids in an auction must all express preferences of a demand type also clarifies the meaning, and the implications, of the restrictions that have been im-

\(^{80}\)Baldwin and Klemperer (in preparation-a) develops the application to matching in detail; preliminary work is in Baldwin and Klemperer (2014). Since our framework allows us to consider multiple players of each kind, it easily yields results along the lines of Chiappori, Galichon, and Salanié (2012).

\(^{81}\)Baldwin, Klemperer, and Milgrom (in preparation). This paper also develops the relationship between the existence of equilibrium for substitutes and properties of the Vickrey auction and the core.

\(^{82}\)The different “goods” were long-term loans (repos) against different qualities of collateral.

\(^{83}\)Any strong substitutes preference could be expressed if the Bank’s “Product-Mix” Auctions (described in Klemperer, 2008, 2010, and Baldwin and Klemperer, in preparation-c) were augmented by permitting “negative” bids (see Klemperer, 2010, and Baldwin and Klemperer, in preparation-c).
posed on the bidders. In particular, the motivation of the Product-Mix Auction is to find competitive equilibrium, given bidders’ and the bid-taker’s reported preferences. Since the Bank of England’s implementation of the Product-Mix auction allows rationing (which makes “goods” divisible) ensuring the existence of equilibrium is not too hard. But in many contexts rationing is less sensible. For example, a too-small piece of radio spectrum may not be useful. Similarly, a government may be interested in offers to build gas-fired plants, nuclear-power stations, wind farms, etc., and these may be indivisible. So results about equilibrium with indivisibilities tell us when Product-Mix Auctions can easily be used.

Our techniques also facilitate the analysis of Product-Mix Auctions. Individuals’ bids in these auctions are aggregated in exactly the same simple way that THs are combined to find aggregate demand. This also makes the auctions more “user-friendly”, and is critical for getting them implemented in practice. Moreover, geometric analysis can develop methods for finding equilibrium in new versions of the Product-Mix Auction; this may help resolve problems currently facing regulators such as the U.S. Federal Communications Commission, the U.K.’s Ofcom and the U.K. Department for Energy and Climate Change.

7 Conclusion

An agent’s demand is completely described by its choices at all possible price vectors. So it can also be described by the *divisions* between the regions of price space in which the agent demands different bundles, and hence by the vectors that define these divisions. This suggests a natural way of classifying valuations into “demand types”.

Using this classification, together with the duality between the geometric representations of valuations in price space and in quantity space, yields significant new insights into when competitive equilibrium exists.

A demand type’s vectors also encode the possible comparative statics of demand, and we expect many other results can be understood more readily, and developed more efficiently, using our geometric perspective.

Companion papers use our framework and tools to obtain new results about the existence of stable matchings in multiple-agent matching models; about individual demand; and further develop the Product-Mix Auction implemented by the Bank of Eng-

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84 Restricting to a demand type also permits relatively complex “bids” while still checking that they satisfy the restrictions, since there are easy software solutions to calculate the normal vectors of the TH for any valuation and so reveal the demand “type”.

85 So the updated (2014) implementation of the Bank’s auction also permitted some complements preferences while maintaining the existence of equilibrium.

86 Baldwin and Klemperer (in preparation-c) shows when equilibrium existence is guaranteed in new forms of the Product-Mix Auction.

87 Product-mix auctions are “one-shot” auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of clock or Simultaneous Multiple-Round Auctions in private value contexts, but they permit the bid-taker to express richer preferences; they are more robust against collusive and/or predatory behaviour; and they are, of course, much faster. (They can also resolve clock auctions’ problem of failing to find the exact equilibrium when it is unique, or the correct equilibrium when it is not.)

88 See Baldwin and Klemperer (in preparation-a, b and c). Preliminary work is in Baldwin and Klemperer (2014).
land in response to the 2007 Northern Rock bank run and the subsequent financial crisis.

References


A Additional formal definitions, and proofs of results in the text

A.1 More details for Section 2

A.1.1 The mathematics and economics of tropical hypersurfaces

Recall we defined the underlying set of a tropical hypersurface associated to valuation $u$ to be

$$
\mathcal{T}_u := \{ p \in \mathbb{R}^n \mid \# D_u(p) > 1 \}.
$$

As stated in the text, a tropical hypersurface has the structure of a weighted rational polyhedral complex. Here we build up that structure by understanding the economic interpretation of its components.

Definition A.1.

(1) The cell interior of the TH $\mathcal{T}_u$ at a price $p$ consists of points $p'$ such that $D_u(p) = D_u(p')$.\(^{89}\) A subset of $\mathcal{T}_u$ is a cell interior if it is the cell interior at some point in $\mathcal{T}_u$.

(2) A subset of $\mathcal{T}_u$ is a cell if it the closure of a cell interior of $\mathcal{T}_u$.

(3) The affine span of a cell of $\mathcal{T}_u$ is the smallest affine space containing the cell.\(^{90}\)

(4) The dimension of a cell is the dimension of its affine span. A cell of dimension $k$ is referred to as a $k$-cell. An $(n - 1)$-cell is referred to as a facet (where $n$ is the number of goods, and so the dimension of Euclidean space in which the TH lives).

(5) The boundary of a cell of $\mathcal{T}_u$ consists of those points in the cell that are not in its cell interior.

(6) A unique demand region (UDR) is a connected component of the complement of the TH.

Note that the cell interior is the largest set that is both contained in the cell and open in the affine span of the cell.\(^{91}\) Naturally, ‘unique demand regions’ are so-called because the demand set contains only one element for such prices.

A TH has the structure of an abstract ‘polyhedral complex’, which we now formally define:

Definition A.2.

(1) A set $\Pi \subseteq \mathbb{R}^n$ is a polyhedral complex if:

(i) $\Pi$ is the union of finitely many cells.

(ii) Each cell is a closed convex polyhedral set in $\mathbb{R}^n$ (that is, each cell may be represented as an intersection of half-spaces $\{ p \in \mathbb{R}^n \mid p \cdot w \geq \alpha \}$ for some vector $w$ and scalar $\alpha$).

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\(^{89}\)Note that cells are subsets of the TH $\mathcal{T}_u$, and not, as one might intuitively guess from looking at Fig. 1, the open areas around the sides of the TH: those are the ‘unique demand regions’.

\(^{90}\)Recall that an affine space in $\mathbb{R}^n$ is a parallel shift of a linear subspace, that is, a set $\{ v + c \mid v \in U \}$ for some linear subspace $U \subseteq \mathbb{R}^n$ and some fixed vector $c$.

\(^{91}\)See the equations for the three objects, given below. One might strictly refer to the ‘cell interior’ as the relative interior of the cell.
(iii) The interiors of the cells do not intersect.
(iv) The boundary of a \(k\)-cell is the union of a finite number of \((k - 1)\)-cells.

(2) \(\Pi\) is a rational polyhedral complex if the slope of the affine span of each cell is rational. That is, in 2., the vectors \(w\) may be taken to have integer coefficients.

(3) \(\Pi\) is \(k\)-dimensional if it is contained in the union of its \(k\)-cells.

It is easy to see that any TH is an \((n - 1)\)-dimensional rational polyhedral complex. Properties 1(i) and 1(iii) follow by definition of the cells. Recognising that cell interiors are defined by a collection of equalities \(p.(x - x') = u(x) - u(x')\) and inequalities \(p.(x-x') > u(x) - u(x')\), and that a cell is defined by weakening these strict inequalities to weak inequalities, property 1(ii) and rationality of the complex (property 2) follow. The boundary of a cell is where at least one of the weak inequalities holds with equality; when this is the case the price must lie in a lower dimensional cell, so property 1(iv) is satisfied. Finally, we note that it is generic for only one bundle to be demanded, and hence the UDRs are \(n\)-dimensional; moreover, they are also polyhedral sets. As the TH is the complement in \(\mathbb{R}^n\) of the UDRs, it follows that the TH is \((n - 1)\)-dimensional.

It is useful to note the economic meaning of cells versus cell interiors:

\textbf{Lemma A.3.} Let \(C\) be a cell, let \(C^\circ\) be its (relative) interior, and fix \(p^\circ \in C^\circ\). Then \(D_u(p^\circ) \subseteq D_u(p)\) iff \(p \in C\), with equality holding iff \(p \in C^\circ\).

In particular, if \(C\) is a 0-cell then \(C^\circ = C\).

\textbf{Proof of Lemma A.3.} As seen above, a cell interior is defined by a collection of equalities and strict inequalities, which are in turn defined the agent’s strict preference for the bundles in the relevant demand set. So these all hold for a price \(p\) iff that price is in the cell interior. The cell itself is defined by replacing all these inequalities by their weak counterpart; these are satisfied iff the agent weakly prefers these bundles to all others, that is, they are contained in the demand set. \(\square\)

For completeness we re-iterate here Definition 2.1 from the text (see there for further discussion of the interpretation of weights):

\textbf{Definition A.4} (Mikhalkin, 2004, Example 2). The tropical hypersurface \(T_u\) associated with any valuation \(u\) is the weighted rational polyhedral complex such that:

(1) its underlying set is \(\{ p \in \mathbb{R}^n | \#D_u(p) > 1\}\);
(2) the weight \(w_u(F)\) of the facet \(F\) is the integer defined by \(w_u(F)v_F = x' - x\), in which \(x'\) is demanded in the UDR on one side of \(F\); \(x\) is demanded in the UDR on the other side; and \(v_F\) is the primitive integer normal vector pointing from the former to the latter.

\textbf{A.1.2 Concavity of valuation functions: Proofs for Section 2.3}

\textbf{Proof of Lemma 2.5.} Suppose that \(A\) is discrete-convex. It is a standard application of the supporting hyperplane theorem that a function \(u : A \rightarrow \mathbb{R}\) is concave iff, for all \(x \in A\), there exists \(p \in \mathbb{R}^n\) such that \(x \in D_u(p)\). As we have defined concavity to require discrete-convexity of domain, this provides the first equivalence. Moreover,
since the intersection between a supporting hyperplane and a convex set is always itself convex, discrete-convexity of \( D_u(p) \) also follows.

Conversely, suppose \( D_u(p) \) is discrete-convex for all \( p \). Let \( u' : A' \to \mathbb{R} \) be the minimal weakly-concave function everywhere weakly greater than \( u \), where \( A' \) is the minimal discrete-convex set containing \( A \). Consider any \( x \in A' \). By the previous equivalence, there exists \( p \) such that \( x \in D_{u'}(p) \). As the minimal weakly-concave function on \( \mathbb{R}^n \) extending \( u \) and \( u' \) on \( \mathbb{R}^n \) must coincide, it follows that \( \text{Conv}_{\mathbb{R}} D_u(p) = \text{Conv}_{\mathbb{R}} D_{u'}(p) \). But by assumption it follows that \( x \in D_u(p) \). So the second property holds (and in particular \( A \) is discrete-convex). □

**Proof of Lemma 2.6.** First see that, for any bundle \( x \), we have \( D_u(p) = \{x\} \) iff \( D_{u'}(p) = \{x\} \), by minimality of \( u' \). So \( D_u(p) \) is single valued iff \( D_{u'}(p) \) is, and at such points the demand sets coincide. Hence both the underlying sets and the weights of the THs coincide. □

### A.1.3 Duality: proofs for Section 2.4

Here we build up the key results for duality in more detail.

Given \( u : A \to \mathbb{R} \) with finite \( A \subseteq \mathbb{Z}^n \), we let \( f_u : \text{Conv}_{\mathbb{R}} A \to \mathbb{R} \) be the minimal weakly-concave function on \( \text{Conv}_{\mathbb{R}} A \) which is everywhere weakly greater than \( u \). Recall that we defined the ‘roof’ of \( u \) to be the graph of \( f_u \).

**Lemma A.5.** The ‘roof’ is a polyhedral complex.

**Proof.** It is clear that the roof is the upper (with respect to the final coordinate) boundary of the convex hull of the points \((x, u(x))\). It is standard (see e.g. Grünbaum and Shephard, 1969) that this has the structure of a polyhedral complex. □

We also now formally define the SNP:

**Definition A.6.** The **subdivided Newton Polytope (SNP)** associated to a valuation \( u : A \to \mathbb{R} \) is the set \( \text{Conv}_{\mathbb{R}} A \) with the structure of a rational polyhedral complex whose cells are the projections to the first \( n \) coordinates of the cells of Lemma A.5.

Again, the dimension of an SNP cell is the dimension if its affine span, and we refer to \( k \)-cells of the SNP in the same way as the TH. (However, a TH may only have cells in dimensions 0 to \( n-1 \), whereas an SNP may have cells in dimensions 0 to \( n \)).

To understand the SNP further we first show:

**Lemma A.7.** For every \( p \in \mathbb{R}^n \), we have \( \text{Conv}_{\mathbb{R}} D_u(p) = D_{f_u}(p) \).

**Proof.** We assume \( f_u \) is weakly concave, so the set bounded above by its graph is convex. By the supporting hyperplane theorem, for every \( x \in \text{Conv}_{\mathbb{R}} A \) there exists a supporting hyperplane passing through \( x \). The result now follows by minimality of \( u \). □

We can now see clearly the economic meaning of the SNP, giving an alternative route to defining it:

**Lemma A.8.** A subset \( \sigma \subseteq \text{Conv}_{\mathbb{R}} A \) is a cell of the SNP iff it has the form \( \text{Conv}_{\mathbb{R}} D_u(p) \) for some \( p \in \mathbb{R}^n \).
Proof. Follows from Lemma A.7. 

As in Lemma A.8, we will always use Greek letters to refer to the cells of an SNP, to distinguish them from the cells of a TH.

Thus SNP cells are naturally associated with convex hulls of demand sets. The relationship between demand for bundles and SNP cells was stated in Lemma 2.9, which we prove first:

**Proof of Lemma 2.9.** We show that, if there exists any price vector \( p' \) such that \( x \in D_u(p') \) then \( x \in D_u(p) \).

For all \( x^\beta \in D_u(p) \), we know \( u(x) - px \leq u(x^\beta) - px^\beta \), with equality only if \( x \in D_u(p) \). So if \( x \in \text{Conv} \ D_u(p) \), i.e., \( x = \sum_\beta \lambda_\beta x^\beta \) for some \( \lambda_\beta \in [0, 1] \) with \( \sum_\beta \lambda_\beta = 1 \), then it follows that \( u(x) - px = \sum_\beta \lambda_\beta (u(x) - px) \leq \sum_\beta \lambda_\beta u(x^\beta) - \sum_\beta \lambda_\beta px^\beta = \sum_\beta \lambda_\beta u(x^\beta) - px \) and so, simplifying, that \( u(x) \leq \sum_\beta \lambda_\beta u(x^\beta) \), with equality only if \( x \in D_u(p) \).

Now suppose \( x \in D_u(p') \). Then \( u(x) - p'x \geq u(x^\beta) - p'x^\beta \) for all \( x^\beta \) so we similarly show that \( u(x) \geq \sum_\beta \lambda_\beta u(x^\beta) \). Hence, if \( x \in D_u(p') \) for any \( p' \), then \( x \in D_u(p) \).

Now recall Proposition A.10. If \( x \in \sigma \) then either \( x \) is demanded for no price, or \( x \in D_u(p) \) for every \( p \in C_\sigma \). \( \square \)

To set up the full relationship between TH cells and SNP cells, we can now conclude

**Corollary A.9.** For any valuation \( u \) and any \( p, p' \in \mathbb{R}^n \), we have \( \text{Conv}_\mathbb{R} D_u(p) = \text{Conv}_\mathbb{R} D_u(p') \iff D_u(p) = D_u(p') \).

**Proof.** From Lemma 2.9, if \( x \in D_u(p') \subseteq \text{Conv}_\mathbb{R} D_u(p') = \text{Conv}_\mathbb{R} D_u(p) \) then \( x \in D_u(p) \). The result follows. \( \square \)

The duality between the SNP and the TH is stated in full as follows

**Proposition A.10** (this extends Lemma 2.8 from the body text). There is a 1-1 correspondence between cells \( \sigma \) of the SNP and the set encompassing both all cells and all unique demand regions of the TH, \( C_\sigma \), such that:

1. \( \sigma = \text{Conv}_\mathbb{R} D_u(p) \) for all \( p \in C_\sigma \);
2. \( C_\sigma = \{ p \in \mathbb{R}^n \mid \sigma \subseteq \text{Conv}_\mathbb{R} D_u(p) \} \);
3. inclusions reverse: \( \sigma \subseteq \sigma' \iff C_{\sigma'} \subseteq C_\sigma \);
4. dimensions are dual: \( \dim \sigma + \dim C_\sigma = n \);
5. cells are orthogonal: \( (p' - p). (x' - x) = 0 \) for all \( p, p' \in C_\sigma, x, x' \in \sigma \).

**Proof.** By Definition A.1.1 the demand set is constant in a cell interior, and by Lemma A.8 every cell \( \sigma \) can be associated to some price \( p \) such that \( \sigma = \text{Conv}_\mathbb{R} D_u(p) \). So (1) gives a well-defined correspondence between an SNP cell \( \sigma \) on the one hand, and a set \( C_\sigma \) which is either a cell interior or a unique demand region. Next, recall from Lemma A.3 that a price \( p \) is in the cell \( C_\sigma \) iff \( D_u(p^\sigma) \subseteq D_u(p) \), where \( p^\sigma \) is some representative element of \( C_\sigma \). It follows by Corollary A.9 that \( p \in C_\sigma \) iff \( \sigma \subseteq \text{Conv}_\mathbb{R} D_u(p) \), i.e. (2) holds. Now (3) follows from the combination of (1) and (2).

For (4) note that the affine span of \( C_\sigma \) is given by the set of prices \( p' \) such that \( u(x) - px = u(x') - px' \) for all \( x, x' \in D_u(p) \), i.e. all prices such that \( p'(x - x') =
u(x) − u(x') for all such x, x'. If σ = Conv\(D_u(p)\) is k-dimensional, these equations impose k linearly independent constraints on such \(p'\), so \(\dim C_\sigma = n - k\).

Moreover, it follows now that \((p'' - p')(x' - x) = 0\) for all \(x, x' \in D_u(p)\) and all \(p', p'' \in C_\sigma\). Thus this equality holds for any \(x, x' \in \text{Conv}_\mathbb{R}D_u(p) = \sigma\), proving (5). □

Note that \(\text{Conv}_\mathbb{R}D_u(p) \neq \sigma\) for \(p\) which are in the boundary of \(C_\sigma\) but not in its interior.

In Section 5 we will be particularly interested in 0-cells of the TH. We will show there that we can check for certain properties by only studying those isolated points. So it is useful to know:

**Corollary A.11.** Given \(u : A \rightarrow \mathbb{R}\), if Conv\(\mathbb{R}A\) is \(n\)-dimensional, then every \(k\)-cell \(C_\sigma\) of \(T_u\) has some 0-cell \(C_\tau\) in its boundary, with \(\sigma \subseteq \tau\). Moreover if \(x \in \sigma\) but \(x \notin D_u(p_\sigma^\tau)\) for \(p_\sigma^\tau \in C_\sigma\), then also \(x \notin D_u(p_\tau^\sigma)\) for \(p_\tau^\sigma \in C_\tau\).

**Proof.** This is easy to see using the SNP: \(\sigma\) is an \((n - k)\)-cell, and by assumption the SNP itself is \(n\)-dimensional, which means that \(\sigma\) is contained in an \(n\)-cell \(\tau\) of the SNP. So there exists a 0-cell \(C_\tau\) of the TH with \(\sigma \subseteq \tau\) by construction and with \(C_\tau \subseteq C_\sigma\) by Proposition A.10.3. That such \(x \notin D_u(p_\tau^\sigma)\) follows from Lemma 2.9. □

### A.1.4 Examples for Section 2.5

**Example A.12.** For a fixed \(A\), it is easy to draw every possible SNP and so obtain every possible combinatorial type of TH, thus enumerating all possible “essentially-different” structures of demand. We do this for \(A = \{0, 1\}^2\) in Fig. 9.

![SNPs](image-url)

Figure 9: All the possible SNPs, and examples of their corresponding combinatorial types of TH when \(A = \{0, 1\}^2\).

It is not hard to see that Fig. 9a applies when \(u(0, 0) + u(1, 1) < u(1, 0) + u(0, 1)\), so represents substitutes; Fig. 9b applies when \(u(0, 0) + u(1, 1) = u(1, 0) + u(0, 1)\), so is additively separable demand; and Fig. 9c applies when \(u(0, 0) + u(1, 1) > u(1, 0) + u(0, 1)\), so is complements. (See Section 3.2 for these distinctions). Importantly, it is clear that these are the only possibilities.
Observe that Fig. 9b can be seen as a limit of Fig. 9a (or, equivalently, Fig. 9c). In the TH, the two 0-cells become arbitrarily close and then coincide in the limit; in quantity space, the faces of the “roof” tilt until they are coplanar, meaning that the SNP edge distinguishing them disappears.

Likewise, any SNP in which the subdivision is not maximal (that is, additional valid \((n - 1)\)-faces could be added) can be recovered by deleting \((n - 1)\)-faces from some SNP whose subdivision is maximal; the corresponding TH is a limit (or ‘degeneration’). Even for larger domains than \(A = \{0, 1\}^2\), we can efficiently enumerate all those combinatorial types of demand for which the SNP subdivision is maximal, knowing we can recover the remainder as their limits:

**Example A.13.** For \(A = \{0, 1, 2\} \times \{0, 1\}\), we list the maximal subdivisions which correspond to THs in Fig. 10.

Figure 10: All the possible SNPs with maximal subdivision, and examples of their corresponding combinatorial types of TH, when \(A = \{0, 1\} \times \{0, 1, 2\}\).

### A.2 Proofs for Section 3: demand types

#### A.2.1 Proofs for Section 3.2: comparative statics

**Proof of Proposition 3.3.** If a valuation \(u\) is not of such a demand type, it must have a facet \(F\) with normal \(v\) where \(v_i, v_j < 0\) for some \(i \neq j\). Then \(e^i \cdot v \neq 0\), i.e., this coordinate vector is not parallel to the facet. So we may choose UDR prices \(p, p' = p + \epsilon e^i\) on either side of the facet. We know demand change from \(p\) to \(p'\) is an integer multiple \(w\) of \(v\). The price for good \(i\) has gone up, so by the law of demand, demand for good \(i\) must have gone down: \(w > 0\). Hence demand for good \(j\) also decreases: goods \(i\) and \(j\) are not substitutes.

Conversely, suppose the valuation \(u\) is of such a demand type. Choose prices \(p' \geq p\) which both lie in unique demand regions. The straight line \([p, p']\) from the first price to the second need not cross only facets, but because the UDRs are open we can choose a small translation vector \(w\) such that \(p + w\), and \(p' + w\) are both respectively in the same unique demand regions as \(p, p'\) and such that \([p + w, p' + w]\) does only cross facets. Let \(x^0, \ldots, x^l\) be demanded in each UDR that this line meets (so in particular \(x^0 = D_u(p)\) and \(x^l = D_u(p')\)). Now, in every case, \(x^i - x^{i-1}\) is an integer multiple
of one of our allowed facet normals, and so has at most one positive and at most one
negative coordinate entry. By the law of demand, demand must weakly decrease at each
step for any good whose price is increasing, and hence demand must weakly increase at
each step for any other good. As the overall change in demand is just the composition
of such changes, this holds for the change in demand from \( x^0 \) to \( x^l \).

Proof of Proposition 3.5 This proof is completely analogous to that of Proposition
3.3. In the first step, we need only suppose that there exists a facet whose normal
has one positive and one negative coordinate entry, and find that these goods are not
complements. In the second step, we see that because demand weakly decreases at each
step for any good whose price is increasing, it must also weakly decrease for all other
goods (since facet normals have the same sign).

The pattern of proof for these two propositions signals that there is potential for
generalisation here. In Baldwin and Klemperer (2014, in preparation-b) we develop
these ideas much more broadly, defining ‘\( D \)-steps’ and showing how they greatly facilitate
study of the comparative statics of demand.

A.2.2 Proofs for Section 3.3: aggregate demand

Suppose we have agents \( j = 1, \ldots, m \) with valuations \( u^j : A_j \to \mathbb{R}^n \). Define their
aggregate domain to be \( A := \sum_{j=1}^m A_j \) and their aggregate valuation to be \( U(y) :=
\max \left\{ \sum_{j=1}^m u^j(x^j) \mid x^j \in A_j, \sum_{j=1}^m x^j = y \right\} \).

Proposition A.14. With agents and aggregate demand as above,

1. \( D_U(p) = \sum_{j=1}^m D_{u^j}(p) \) for all \( p \in \mathbb{R}^n \).
2. \( T_U \) has underlying set equal \( \bigcup_{j=1}^m T_{u^j} \), and the weight on any facet of \( T_U \) is equal
to the sum of weights of individual \( TH \) facets that contain it.

Proof. (1) For any \( p \in \mathbb{R}^n \), note that

\[
\sum_{j=1}^m D_{u^j}(p) = \sum_{j=1}^m \max_{x^j \in A_j} \left\{ u^j(x^j) - p.x^j \right\} = \max \left\{ \sum_{j=1}^m u^j(x^j) - p.\left( \sum_{j=1}^m x^j \right) \mid x^j \in A_j \right\} ,
\]

and on the other hand (since \( y \in A \) iff \( y = \sum_{j=1}^m x^j \) where \( x^j \in A_j \)) that

\[
\max_{y \in A} \{ U(y) - p.y \} = \max \left\{ \max \left\{ \sum_{j=1}^m u^j(x^j) \mid x^j \in A_j, \sum_{j=1}^m x^j = y \right\} - p.y \mid y = \sum_{j=1}^m x^j, x^j \in A_j \right\} = \max \left\{ \sum_{j=1}^m u^j(x^j) - p.\left( \sum_{j=1}^m x^j \right) \mid x^j \in A_j \right\} ,
\]

and that the same arguments \( x^j \in A \), with \( y = \sum_{j=1}^m x^j \), are maximising in either case.

(2) By (1), \( D_U(p) \) is single-valued iff \( \sum_{j=1}^m D_{u^j}(p) \) is single-valued, and hence iff \( D_{u^j}(p) \) is single-valued for all \( j \). Thus the underlying sets given coincide. Suppose \( F \)
is a facet of $T_U$ with adjacent UDRs $R$ and $R'$; let $v_F$ be a primitive integer vector pointing from $R$ to $R'$. For $j = 1, \ldots, m$, write the demand of agent $j$ in $R$ as $x^j$ and in $R'$ as $x'^j$ (for some agent these will be distinct, but not necessarily for all). Then $w_{\omega}(F)v_F = x'^j - x^j$ for all $j$, and so

$$
\sum_j w_{\omega}(F)v_F = \sum_j x'^j - \sum_j x^j = w_U(F)v_F,
$$

as required. □

A.3 Proofs for Section 4: Equilibrium

A.3.1 Proof of Theorem 4.2

First, it is useful to know equivalent characterisations of unimodularity:

Remark A.15. The following are equivalent, for a set of $s$ linearly independent vectors in $\mathbb{Z}^n$:

1. They are an integer basis for the subspace they span.
2. A $s$-dimensional parallelepiped in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$ and these vectors as edges contains no point in $\mathbb{Z}^n$ except its vertices;
3. they can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant $\pm 1$;
4. among the determinants of all the $s \times s$ matrices consisting of $s$ rows of the $n \times s$ matrix whose columns are these $s$ vectors, the greatest common factor is $1$.

Proofs of these facts may be found in Cassels (1971). We refer to a set of vectors as unimodular if every linearly independent subset has these properties.

As described in Section 4.2, we first prove necessity and sufficiency for transverse intersections, and then show that the general case follows. We start with necessity:

Lemma A.16. Consider $s \leq n$ agents each of whose demand set includes precisely 2 bundles at price $p$, i.e., $\#D_u(i) = 2$, for $i = 1, \ldots, s$. Write $v^i$ for the difference between the two bundles demanded by agent $i$ (so $v^i$ is normal to $i$’s facet of demand at $p$). Suppose the $s$ vectors $v^1, \ldots, v^s$ are linearly independent. Write $U$ for the aggregate valuation. There exists an integer bundle in Conv $D_U(p)$ which is not demanded at any price iff vectors $v^1, \ldots, v^s$ do not form a unimodular set.

Proof. By Lemma 2.9, an integer bundle in Conv $D_U(p)$ is not demanded at any price iff it is not in $D_U(p)$. Now, each individual agent $i$’s demand at $p$ has the form

\footnote{This is made completely precise in Fact A.40 below.}

\footnote{This fact is especially helpful when developing examples.}

\footnote{(1)$\iff$(3) follows from Cassels (1971) Lemma 1.1 and Corollary 1.3. (1)$\iff$(4) is Cassels (1971) Lemma 1.2. For (1)$\iff$(2) consider a parallelepiped $P$ whose vertices are $y + \sum_{i=1}^s a_i w^i$ for $a_i \in \{0, 1\}$. If $z$ is a non-vertex integer point in $P$, then $z - y$ exhibits the failure of (1). Conversely, if failure of (1) is exhibited by an integer $\sum_{i=1}^s b_i w^i$ where $b_i$ are not all integers, then $y + \sum_{i=1}^s a_i w^i$ exhibits failure of (2), where $a_i$ is the non-integer part of $b_i$ in each case.}
\[ D_u(p) = \{ y^i + \delta_i v_i \mid \delta_i \in \{0, 1\} \}, \] where \( y^i \) is the bundle demanded on the appropriate side of the TH facet. So the set of bundles demanded on aggregate at \( p \) is

\[ D_U(p) = \{ y + \delta_1 v^1 + \cdots + \delta_s v^s \mid \delta_i \in \{0, 1\}, i = 1, \ldots, s \}, \]

where \( y = \sum_i y^i \). These points are precisely the vertices of an \( s \)-dimensional parallelepiped in \( \mathbb{Z}^n \) (since its edges, the \( v_i \), are linearly independent). There exists an integer bundle in \( \text{Conv} D_U(p) \) which is not in \( D_U(p) \) iff this parallelepiped contains an integer bundle which is not a vertex, and, by Remark A.15.2 and A.15.3, this holds iff the set \( \{ v^1, \ldots, v^s \} \) is not unimodular. \( \square \)

Next, sufficiency:

**Proposition A.17.** Suppose price \( p \) is in the interior of an \( (n - k_i) \)-cell \( C_i \) of the TH \( T_{u^i} \) of each of \( s \) agents \( i = 1, \ldots, s \), who have concave valuations \( u^i \), and together have aggregate valuation \( U \). Then every integer bundle in \( \text{Conv} D_U(p) \) is demanded at \( p \) if each \( C_i \) is a subset of the intersection of a set of facets \( F^1_i, \ldots, F^k_i \) of \( T_{u^i} \) (not necessarily comprising all facets of \( T_{u^i} \) that pass through \( C_i \)) with primitive integer normal vectors \( v_1^i, \ldots, v_s^i \) and \( \{ v_j^i \mid i = 1, \ldots, s; j = 1, \ldots, k_i \} \) are unimodular.

**Proof.** All bundles demanded by agent \( i \) at \( p \) are demanded throughout the \( (n - k_i) \)-cell \( C_i \), which corresponds to a \( k_i \)-dimensional polytope \( \sigma_i \) in the SNP of agent \( i \). Moreover, \( \sigma_i \) possesses an edge in direction \( v_j^i \) for \( j = 1, \ldots, k_i \); each corresponds to the facet \( F_j^i \). Thus, if \( y^i \) is some integer bundle in \( D_{u^i}(p) \), then (by a dimension count) the affine span of \( \sigma_i \) is precisely \( \{ y^i + \sum_{j=1}^{k_i} \beta_j v_j^i \mid \beta_j \in \mathbb{R} \} \) for \( j = 1, \ldots, k_i \}, \) and in particular, \( D_{u^i}(p) \) is contained in this set.

Thus, since aggregate demand is the Minkowski sum of individual demand, we may express aggregate demand among these agents as

\[ D_U(p) = \{ y + \sum_{i=1}^s \sum_{j=1}^{k_i} a_{ij}^i v_j^i \mid y^i + \sum_{j=1}^{k_i} a_{ij}^i v_j^i \in D_{u^i}(p) \text{ for } i = 1, \ldots, s \} \]

where \( y := \sum_{i=1}^s y^i \).

Now, suppose \( x \) is an integer bundle in \( \text{Conv} D_U(p) \). Then \( x - y \) is in the span of the \( v_j^i \). But since they are an integer basis for their span, we can write \( x - y = \sum_{i=1}^s \sum_{j=1}^{k_i} b_{ij} v_j^i \), for some \( b_{ij} \in \mathbb{Z} \). So we can define \( x^i := y^i + \sum_{j=1}^{k_i} b_{ij} v_j^i \), and know that \( x^i \in \mathbb{Z}^n \).

But we also know \( x^i \in \text{Conv} D_{u^i}(p) \). To see this, observe that since \( x \in \text{Conv} D_U(p) \), we can write \( x - y = \sum_{i=1}^s \sum_{j=1}^{k_i} a_{ij}^i v_j^i \) for some finite set of weights \( \lambda_{ij} \in [0, 1] \) such that \( \sum_{i=1}^s \sum_{j=1}^{k_i} \lambda_{ij} = 1 \) and such that \( x^i = y^i + \sum_{j=1}^{k_i} a_{ij}^i v_j^i \in D_{u^i}(p) \) for each agent \( i \) and for each \( j \). But since the \( v_j^i \) are linearly independent, there is an unique way to write \( x - y \) as a weighted sum of the \( v_j^i \), so \( b_{ij} = \sum_{i=1}^s \lambda_{ij} a_{ij}^i \), and so \( x^i = y^i + \sum_{j=1}^{k_i} b_{ij} v_j^i = y^i + \sum_{j=1}^{k_i} \sum_{i=1}^s \lambda_{ij} a_{ij}^i v_j^i \in \text{Conv} D_{u^i}(p) \).

So \( x^i \) is an integer vector in \( \text{Conv} D_{u^i}(p) \). By concavity of \( u^i \) there exists some price at which \( x^i \) is demanded by agent \( i \) (Lemma 2.5), and so by Lemma 2.9 we know \( x^i \in D_{u^i}(p) \). Thus \( x = \sum_{i=1}^s x^i \in D_U(p) \). That is, \( x \) is demanded at \( p \), as required. \( \square \)

Finally we deal with the non-transverse case. Start by recalling Proposition 4.6: given valuations \( u^1 \) and \( u^2 \), for generic \( v \in \mathbb{R}^n \) and small enough \( \epsilon \), the intersection \( (T_{u^1}) \cap (\epsilon v + T_{u^2}) \) is transverse.
Strictly speaking, we should have noted that $v + T_n$ is also a TH to state this result, although it is obviously a balanced weighted rational polyhedral complex of dimension $n - 1$, so we can apply Theorem 2.3. However, in the following we will need to know explicitly the corresponding valuation:

**Lemma A.18.** Let $u : A \to \mathbb{R}$ be a valuation and let $\epsilon > 0$ and $v \in \mathbb{R}^n$. Define $u_\epsilon : A \to \mathbb{R}$ by $u_\epsilon(x) = u(x) + \epsilon v \cdot x$. Then:

1. $D_{u_\epsilon}(p) = D_u(p - \epsilon v)$ for all $p \in \mathbb{R}^n$;
2. $T_{u_\epsilon} = \epsilon v + T_u$;
3. $\|u_\epsilon(x) - u(x)\| \leq R\|v\|$, where $R$ satisfies $R > \|x\|$ for all $x \in A$.

**Proof.** First see

$$D_{u_\epsilon}(p) = \arg\max_{x \in A} \{u(x) + \epsilon x \cdot v - x \cdot p\} = \arg\max_{x \in A} \{u(x) - x \cdot (p - \epsilon v)\} = D_u(p - \epsilon v).$$

The remainder of the lemma follows by definition of $T_u$, and the Cauchy-Schwarz inequality.

Now we relate this material to the question of competitive equilibrium:

**Lemma A.19.** Suppose we have agents 1, 2 with valuations $u_1^1, u_2^1$ and supply $x \in A_1 + A_2$ for which competitive equilibrium does not exist: there does not exist $p$ such that $x \in D_U(p)$. Then for any $v \in \mathbb{R}^n$, competitive equilibrium also fails when the agent's valuations are $u_1^2$ and $u_2^2$, for all sufficiently small $\epsilon$.

**Proof.** Let $p$ be a price such that $x \in \text{Conv } D_U(p)$, $x \notin D_U(p)$. (Such a price exists since the 'SNP' subdivision subdivides the whole of Conv$_\mathbb{R} A$).

Use indices $\beta$ to label the elements of $D_U(p)$. By assumption there exist $\lambda_\beta \in [0, 1]$ with $\sum_\beta \lambda_\beta = 1$ such that $x = \sum_\beta \lambda_\beta y^\beta$. Following the logic of Section 2.4 it must be that $U(x) < \sum_\beta \lambda_\beta U(y^\beta)$. So pick $\eta$ so that

$$U(x) < \sum_\beta \lambda_\beta U(y^\beta) - \eta. \tag{1}$$

Now, fix any $v \in \mathbb{R}^n$. Let $\epsilon > 0$ be any real satisfying $\epsilon \leq \frac{\eta}{2\|v\|}$, where $R > \|x\|$ for all $x \in A_2$, the domain of $u^2$. Let $u_\epsilon^2$ be as given in the statement of the lemma and let $U_\epsilon$ be the aggregate valuation of $u_1^1$ and $u_\epsilon^2$. Now, there exist $x^1 \in A_1$, $x^2 \in A_2$ such that

$$U_\epsilon(x) = u_1^1(x^1) + u_\epsilon^2(x^2) = u_1^1(x^1) + u_2^1(x^2) + \epsilon v \cdot x^2 \leq U(x) + \epsilon v \cdot x \leq U(x) + \frac{1}{2}\eta$$

by Lemma A.18 and definition of $\epsilon$. Since we can apply Lemma A.18 again, reversing the roles of $u_2^1$ and $u_\epsilon^2$ (and reversing the sign of $v$), it follows that $\|U_\epsilon(x) - U(x)\| \leq \frac{1}{2}\eta$. Moreover, we can apply the same argument for every $y^\beta \in D_U(p)$, so $\|U_\epsilon(y^\beta) - U(x^\beta)\| \leq \frac{1}{2}\eta$ for all $\beta$. It follows from Equation (1) that $U_\epsilon(x) < \sum_\beta \lambda_\beta U(y^\beta)$. By definition of the $\lambda_\beta$, we conclude that $U_\epsilon$ is not concave at $x$, and so that $x$ is never aggregate demand when the aggregate valuation is $U_\epsilon$, which completes the proof. \qed
Proposition A.20. If competitive equilibrium fails for a set of agents, then there exists a small perturbation of their valuations such that equilibrium still fails and such that all intersections are transverse.

Proof. This follows from repeated application of Proposition 4.6 and Lemma A.19. □

Thus, to analyse when competitive equilibrium exists for all agents of a given type, we need only be concerned with those whose THs meet transversally wherever they intersect. Hence:

Proof of Theorem 4.2. Necessity of unimodularity of the demand type is given by Proposition A.16. Sufficiency is is give by Proposition A.17 and the contrapositive of Proposition A.20. □

A.3.2 Proofs for Section 4.3.2: Basis changes

Proposition A.21 (cf. Gorman, 1976, p. 219-20). For $A \subseteq \mathbb{Z}^n$ and $u : A \rightarrow \mathbb{R}$ and a unimodular $n \times n$ matrix $G$, define the (“pullback”) basis change of $u$ by $G$ to be $G^*u : G^{-1}A \rightarrow \mathbb{R}$ via $G^*u(y) := u(Gy)$. Then

(1) A bundle is demanded under the original demand at a certain price iff an associated bundle is demanded under the transformed demand at an associated price; specifically: $x \in D_u(p) \iff G^{-1}x \in D_{G^*u}(G^Tp)$.

(2) The TH of the transformed demand is given by a linear transformation of the original demand: $\Gamma_{G^*u} = \{G^T \mathbf{p} \mid \mathbf{p} \in \Gamma_u\}$.

(3) The inverse transformation to $G$ applies to demand types: $u(\cdot)$ is of (concave) demand type $\mathcal{D}$ iff $G^*u(\cdot)$ is of (concave) demand type $G^{-1}\mathcal{D} = \{G^{-1}v \mid v \in \mathcal{D}\}$.

Proof. 1. By definition, $x \in D_u(p)$ if $p^T(x - x') \leq u(x) - u(x')$ for all $x' \in A$, with equality iff $x' \in D_u(p)$ also. For any invertible matrix $G$, we may re-write

$$p^T(x - x') = p^TGG^{-1}(x - x') = (G^Tp)^T(G^{-1}x - G^{-1}x').$$

If $G$ is additionally unimodular, then $G^{-1}x$ and $G^{-1}x' \in \mathbb{Z}^n$. We define a new valuation $G^*u$ on the finite set $G^{-1}A \subseteq \mathbb{Z}^n$ via $G^*u(y) := u(Gy)$. If we write $y = G^{-1}x$ and $y' = G^{-1}x'$ then $(G^Tp)^T(y - y') \leq G^*u(y) - G^*u(y')$ holds iff $p^T(x - x') \leq u(x) - u(x')$. So we have

$$x \in D_u(p) \iff y = G^{-1}x \in D_{G^*u}(G^Tp),$$

as required.

2. Since the underlying set of $\Gamma_u$ is those $p$ for which $\#D_u(p) > 1$ it follows immediately from 1. that $\Gamma_{G^*u} = \{G^T \mathbf{p} \mid \mathbf{p} \in \Gamma_u\}$, as required.

3. Suppose $v$ is normal to a facet $F$ of $\Gamma_u$. It follows from 2. that the facet corresponding to $F$ in $\Gamma_{G^*u}$ has the form $G^TF = \{G^T \mathbf{p} \mid \mathbf{p} \in F\}$. We know $p^Tv$ is constant for $p \in F$, from which it follows that $(G^Tp)^Tv = p^TGG^{-1}v$ is constant for $G^Tp \in G^TF$: we see $G^{-1}v$ is normal to a facet of $\Gamma_{G^*u}$. As $G$ has an integer inverse, the converse is also true. Trivially, for any unimodular matrix $G$, the valuation $G^*u$ is concave iff the valuation $u$ is. □
A.4 Proofs for Section 5: Application of the tropical Bézout-Kouchnirenko-Bernshtein theorem

To provide proofs for the material in Section 5, we need to associate additional mathematical structure to the TH and SNP. In particular, it is possible to re-interpret our previous results on equilibrium (Propositions A.16 and A.17) in terms of subgroup indices of lattices associated with SNPs (see Proposition A.28 below). These subgroup indices are used in the definition of intersection multiplicity in tropical geometry, and thus allow us to apply results from tropical intersection theory to the question of competitive equilibrium.

For Appendices A.4.1 and A.4.2 we work in greater generality than required for the proofs of Theorems 5.3, 5.5 and 5.6: we consider intersections of more than two THs. We do not need such results to apply Theorems 5.3, 5.5 and 5.6 to multiple agents. Instead, we should first consider the intersection of the first two agents, then the intersection of the third with the aggregation of the first two, etc. However, it is easy to state the results in full generality, and the relationships between subgroup indices, TH intersections and equilibrium go beyond those theorems. In particular Proposition A.28 provides the key link between economics and geometry, in general.

This Appendix section is followed by Appendix A.5, which explains mathematical terms that may be unfamiliar to some readers, on lattices, volumes and and subgroup indices.\footnote{Note that our use of ‘lattice’ is in its group-theoretic meaning; see Appendix A.5 below. ‘Lattices’ are also used economics in their order-theoretic sense, particularly in work on comparative statics (see e.g. Milgrom and Shannon, 1994). We emphasise that mathematics has unfortunately used the same word with two entirely different meanings.}

We use the following notation throughout this appendix:

**Notation** We suppose that the THs of $r$ agents intersect at some price $p$ (although there may be additional agents in the economy). Label these $r$ agents as $1, \ldots, r$ and let $\mathcal{T}_U$ be the aggregate TH of $\mathcal{T}_{u_1}, \ldots, \mathcal{T}_{u_r}$. Label as $C$ a cell of $\mathcal{T}_U$ in the intersection of these individual THs at $p$. Label the smallest cells containing $C$ of $\mathcal{T}_{u_j}$ as $C_j$, and let $\sigma$, $\sigma_j$ be the respective corresponding SNP cells, for $j = 1, \ldots, r$. Set $d_j := \text{dim } \sigma_j$ and $d := \text{dim } \sigma$. Note that $d_j \geq 1$ for all $j$ because $C_j$ is a cell of the TH (as distinct from a UDR).

A.4.1 Parallel linear spaces for an SNP

To define the relevant lattices, we first provide definitions and results on their associated linear (vector) spaces.

**Definition A.22.** Given an SNP cell $\rho$, the parallel linear space $L_\rho$ is the vector space parallel to the affine span of $\rho$: $L_\rho$ consists of all linear combinations of the vectors \{ $x - y$ | $x, y \in \rho$ \}

Clearly, the dimension of $\rho$ is equal to the dimension of $L_\rho$.

**Lemma A.23.** In the notation of Appendix A.4, $\sigma = \sigma^1 + \cdots + \sigma^r$.\footnote{Note that our use of ‘lattice’ is in its group-theoretic meaning; see Appendix A.5 below. ‘Lattices’ are also used economics in their order-theoretic sense, particularly in work on comparative statics (see e.g. Milgrom and Shannon, 1994). We emphasise that mathematics has unfortunately used the same word with two entirely different meanings.}
Proof. By definition, for any \( p \) in the relative interior of \( C \) (and hence of \( C_j \) for relevant \( j \)) we have \( DU(p) = Du_1(p) + \cdots + Du_r(p) \). So by the properties of polytopes\(^{96}\) we know \( \sigma = \sigma_1 + \cdots + \sigma_r \). \( \square \)

Corollary A.24. In the notation introduced at the beginning of Appendix A.4, \( L_\sigma = L_{\sigma_1} + \cdots + L_{\sigma_r} \).

Proof. Follows from Definition A.22 and Lemma A.23 \( \square \)

Lemma A.25. In the notation introduced at the beginning of Appendix A.4:

1. If \( r = 2 \), the intersection of \( T_{u_1} \) and \( T_{u_2} \) is transverse at \( C \) iff \( L_{\sigma_1} \cap L_{\sigma_2} = \{0\} \).

2. For any \( r \geq 1 \), the intersection of \( T_{u_1}, \ldots, T_{u_r} \) is transverse at \( C \) iff \( L_\sigma = L_{\sigma_1} \oplus \cdots \oplus L_{\sigma_r} \), which holds iff \( d_1 + \cdots + d_r = d \).\(^{97}\)

Proof. (1) By definition of transversality,

\[
\begin{align*}
n &= \dim(C^1 + C^2) = \dim C^1 + \dim C^2 - \dim C \\
&= (n - d^1) + (n - d^2) - (n - d) \\
\iff d^1 + d^2 &= d
\end{align*}
\]

On the other hand, we know from Lemma A.24 that \( L_\sigma = L_{\sigma_1} + L_{\sigma_2} \) and so \( \dim L_\sigma = \dim L_{\sigma_1} + \dim L_{\sigma_2} - \dim(L_{\sigma_1} \cap L_{\sigma_2}) \). Since the dimension of a polytope is defined to be the dimension of its affine span, which is naturally equal to the dimension of the parallel linear space, we conclude that \( L_{\sigma_1} \cap L_{\sigma_2} = \{0\} \) iff \( \dim(C^1 + C^2) = n \), i.e. iff the intersection is transverse at \( C \).

(2) Suppose \( r = 2 \). Then, by part (1), we know \( L_{\sigma_1} \cap L_{\sigma_2} = \{0\} \); we also know that \( L_\sigma = L_{\sigma_1} + L_{\sigma_2} \) by Corollary A.24. So \( L_\sigma = L_{\sigma_1} \oplus L_{\sigma_2} \). For \( r \geq 3 \), recall from Definition 4.5 that we check transversality incrementally, taking the aggregate THs of agents 1, \ldots, \( j \) and checking transversality of the intersection of this with the \( (j + 1) \)th TH. Applying the \( r = 2 \) case each time yields us the first result. Finally, Corollary A.24 shows that \( \sigma \geq d^1 + \cdots + d_r \) in general, and by basic linear algebra, \( \sigma \geq d^1 + \cdots + d^r \) is possible iff \( L_\sigma = L_{\sigma_1} \oplus \cdots \oplus L_{\sigma_r} \). \( \square \)

### A.4.2 Lattices associated with an SNP

Here we give the definitions and results we will need to prove Theorems 5.3, 5.5 and 5.6. Basic material on lattices in this context is presented in Appendix A.5.

**Definition A.26.** Given an SNP cell \( \rho \), the **parallel lattice** \( N_\rho \) is the set of integer vectors parallel to the cell: \( N_\rho = L_\rho \cap \mathbb{Z}^n \).

**Lemma A.27.** In the notation introduced at the beginning of Appendix A.4, the lattice \( N_{\sigma_1} + \cdots + N_{\sigma_r} \) is a sublattice of \( N_\sigma \), and the linear spans coincide.

---

\(^{96}\)See e.g. Cox et al 2005, Section 7.4, Exercise 3.

\(^{97}\)In linear algebra, "\( L = L_1 \oplus L_2 \)" is shorthand for saying that the (Minkowski) sum of vector spaces \( L_1 + L_2 = L \) and also that \( L_1 \cap L_2 = \{0\} \). It extends naturally across several vector spaces.
Proof. By Corollary A.24, for \(j = 1, \ldots, r\), we know \(L_{\sigma_j} \subseteq L_\sigma\). Take intersections with \(\mathbb{Z}^n\) to see \(N_{\sigma_j} \subseteq N_\sigma\). Thus, as \(N_\sigma\) is additively closed, \(N_{\sigma_1} + \cdots + N_{\sigma_r} \subseteq N_\sigma\). It is a sublattice as it is a lattice.

We know \(L_\sigma = L_{\sigma_1} + \cdots + L_{\sigma_r}\), so, to show the linear spans coincide, it is sufficient to show that \(L_{\sigma_1} + \cdots + L_{\sigma_r}\) is the linear span of \(N_{\sigma_1} + \cdots + N_{\sigma_r}\). But the latter must certainly contain \(L_{\sigma_j}\) for \(j = 1, \ldots, r\) and so it contains their sum; on the other hand \(N_{\sigma_1} + \cdots + N_{\sigma_r} \subseteq L_{\sigma_1} + \cdots + L_{\sigma_r}\) and the latter is linear; so \(L_{\sigma_1} + \cdots + L_{\sigma_r}\) is indeed the minimal linear vector subspace of \(\mathbb{R}^n\) containing \(N_{\sigma_1} + \cdots + N_{\sigma_r}\). \(\Box\)

Importantly, we thus have a well-defined subgroup index \([N_\sigma : N_{\sigma_1} + \cdots + N_{\sigma_r}]\) (see Appendix A.5.2). Note from Fact A.45 that this is greater than 1 precisely when the parallelepiped we have discussed before contains a non-vertex point.

Having understood this definition, we now re-write and slightly generalise Propositions A.16 and A.17:

**Proposition A.28.** Use the notation introduced at the beginning of Appendix A.4. Suppose \(p\) is in the relative interior of \(C\).

1. If \([N_\sigma : N_{\sigma_1} + \cdots + N_{\sigma_r}] = 1\) then \(D_U(p)\) is discrete-convex.
2. If \([N_\sigma : N_{\sigma_1} + \cdots + N_{\sigma_r}] > 1\) and if \(\dim \sigma^1 \leq 2\) and \(\dim \sigma^j = 1\) for \(j = 2, \ldots, r\) then \(D_U(p)\) is not discrete-convex.

Proof. Part (1) slightly extends Proposition A.17, to cover cases in which there is no basis for \(N_{\sigma_j}\) consisting of edges of \(\sigma^j\). However, in the proof of that proposition, we only used the fact that the set of vectors we assigned to agent \(j\) was a set of integer vectors and was a basis for the linear span for that agent’s demand set. Thus we need only make the weaker assumption that some integer basis exists for each agent’s parallel lattice so that the combination of the bases is unimodular, to obtain the result in the same way.

Part (2) is identical to Proposition A.16 when \(\dim \sigma^1 = 1\). Suppose \(\dim \sigma^1 = 2\). Without loss of generality we may assume that \(0 \in \sigma^j\) for \(j = 1, \ldots, r\) (otherwise the following arguments are simply augmented by a fixed shift). For \(j = 2, \ldots, r\), fix an minimal integer non-zero vector \(v^j \in \sigma^j\). In each case this vector then forms a basis for the corresponding lattice \(N_{\sigma_j}\).

For \(j = 1\) we will need find a basis for \(N_{\sigma_1}\) consisting of vectors \(v^0, v^1\) which are actually contained inside \(\sigma^1\) (note that this is not immediate). Start by taking \(w^0, w^1 \in \sigma^1\) which are linearly independent integer vectors. If these are a basis for \(N_{\sigma_1}\), we are done. If not, they span a sublattice \(M_1\) of \(N_{\sigma_1}\), such that \([N_{\sigma_1} : M_1] > 1\), and so there must exist \(w \in \mathbb{Z}^n\) which is a non-vertex point of the parallelepiped they span. Then \(w = \alpha^0 w^0 + \alpha^1 w^1\) with \(\alpha^0, \alpha^1 \in [0, 1]\). If \(\alpha^0 + \alpha^1 \leq 1\) then we fix \(w^2 := w\); if \(\alpha^0 + \alpha^1 > 1\) then let \(w^2 = w^0 + w^1 - w\). In either case now \(w^2 = \beta^0 w^0 + \beta^1 w^1\) with \(\beta^0 + \beta^1 \leq 1\). As \(\sigma^1\) is convex we conclude that \(w^2 \in \sigma\).

Recall \(w\) was a non-vertex point of the parallelepiped spanned by \(w^0, w^1\), we know \(w^2 \neq w^0, w^1, 0\). So \(w^2\) is a non-vertex point of the convex hull \(\Delta^0\) of \(0, w^0, w^1\), and hence the convex hull \(\Delta^1\) of \(0, w^1, w^2\) has strictly smaller area than \(\Delta^0\). Moreover the parallelepipeds spanned by \(w^0, w^1\) and \(w^1, w^2\) respectively have areas equal to twice the areas of \(\Delta^0, \Delta^1\), respectively. So, if \(M_2\) is the sublattice of \(N_{\sigma_1}\) spanned by \(w^1, w^2\), then \([N_{\sigma_1} : M_2] < [N_{\sigma_1} : M_1]\).
As all subgroup indices are positive-integer-valued, after a finite number of repetitions of this process, the subgroup index will be 1, and hence (by Fact A.45.3) we will have obtained vectors \( v^0, v^1 \in \sigma^1 \) which are a basis of \( N_\sigma \), as required.

Now, by Fact A.45.3, there exists a bundle \( x \in N_\sigma \), \( x \notin N_{\sigma^1} + \cdots + N_{\sigma^r} \). Our identified vectors \( v^0, v^1, \ldots, v^r \) are an integer basis for this sublattice of \( N_\sigma \). Because the linear spans coincide (by Lemma A.27), they are thus a basis for \( L_\sigma \), so we can write \( x \) as a (real-valued) linear combination of these vectors. Moreover, since subtracting integer multiples of the \( v^j \) from \( x \) yields a new element of \( N_\sigma \), we can assume that \( x \) is in the fundamental parallellepiped of the sublattice with respect to this basis. So we can write \( x = \sum_{j=0}^r \alpha^j v^j \) with \( \alpha^j \in [0, 1] \) for \( j = 0, \ldots, r \). Additionally, we can assume that \( \alpha^0 + \alpha^1 \leq 1 \) (if \( \alpha^0 + \alpha^1 > 1 \) then replace \( x \) with \( \sum_{j=0}^r v^j - x \in N_\sigma \)). Now \( \alpha^0 v^0 + \alpha^1 v^1 \in \sigma^1 \) and, for all \( j \), also \( \alpha^j v^j \in \sigma^j \). So, \( x \in \sigma^1 + \cdots + \sigma^r = \sigma = \text{Conv } D_U(p) \). Moreover, \( x \in N_\sigma \subseteq \mathbb{Z}^n \). But, by assumption, \( x \notin N_{\sigma^1} + \cdots + N_{\sigma^r} \), and so \( x \notin D_{u^1}(p) + \cdots + D_{u^r}(p) = D_U(p) \).

To show the tightness of this result, we now give an example where \( [N_\sigma : N_{\sigma^1} + N_{\sigma^2}] > 1 \), where \( \dim \sigma^1 = \dim \sigma^2 = 2 \) and where \( D_U(p) \) is discrete-convex. Although there will be a non-vertex lattice point in the fundamental parallelepiped of \( N_{\sigma^1} + N_{\sigma^2} \), the key question for us is whether that lattice point is actually in \( \sigma^1 + \sigma^2 \). If both these SNP cells are small enough, for example if they are both simplices, the answer can be ‘no’. But the answer can also be ‘yes’, without any change in the volumes or subgroup indices, and so it seems that our method of analysis reaches its limit here.

**Example A.29.** In this example \( n = 4 \).

Agent 1 has valuation \( u^1(0, 0, 0, 0) = 0, u^1(1, 1, 0, 1) = 6 \), \( u^1(0, 0, 1, 1) = 6 \). So Agent 1 is indifferent between these three bundles for prices \( p \) such that \( p_1 + p_2 = 6, p_3 + p_4 = 6 \). There are three facets, which in this case are 3-dimensional, and on which the agent is indifferent between pairs of these bundles, emanating from this 2-cell.

Agent 2 has valuation \( u^2(0, 0, 0, 0) = 0, u^2(0, 1, 1, 0) = 9, u^2(4, 0, 0, 1) = 6 \), and so is indifferent between these bundles for prices \( p \) such that \( p_2 + p_3 = 9 \) and \( 4p_1 + p_4 = 6 \). Again, there are three facets, on which the agent is indifferent between pairs of these bundles, emanating from this 2-cell.

These conditions are all satisfied iff \( p = (1, 5, 4, 2) \). At this price, we have \( \sigma^1 = \text{Conv } ((0, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 1)) \) and \( \sigma^2 = \text{Conv } ((0, 0, 0, 0), (0, 1, 1, 0), (4, 0, 0, 1)) \). The aggregate SNP cell \( \sigma \) is the convex hull of all these points; it is of course 4-dimensional and so \( N_\sigma = \mathbb{Z}^4 \). Meanwhile \( N_{\sigma^1} \) and \( N_{\sigma^2} \) are 2-dimensional lattices, and we check that the non-zero vectors we already know in each lattice do give a basis in each case, by checking that the sets \( \{(1, 1, 0, 0), (0, 0, 1, 1)\} \) and \( \{(0, 1, 1, 0), (4, 0, 0, 1)\} \) are unimodular (use Remark A.15.4).

Thus, \([N_\sigma : N_{\sigma^1} + N_{\sigma^2}]\) is given by the absolute value of the determinant of these four vectors; it is clearest to see the value of this determinant by re-ordering them:

\[
\begin{vmatrix}
1 & 0 & 0 & 4 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{vmatrix} = -3
\]

This tells us that there are exactly 2 interior points to the fundamental parallellepiped.
of $N_{\sigma_1} + N_{\sigma_2}$. We express them explicitly in terms of the basis vectors:

\[
\begin{pmatrix}
\frac{2}{3} \\
0 \\
0
\end{pmatrix}
+ \frac{2}{3}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
+ \frac{1}{3}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
+ \frac{1}{3}
\begin{pmatrix}
4 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
2 \\
1 \\
1
\end{pmatrix}
\]

(2)

and

\[
\begin{pmatrix}
\frac{1}{3} \\
0 \\
0
\end{pmatrix}
+ \frac{1}{3}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
+ \frac{2}{3}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
+ \frac{2}{3}
\begin{pmatrix}
4 \\
0 \\
1
\end{pmatrix}
= \begin{pmatrix}
3 \\
1 \\
1
\end{pmatrix}
\]

(3)

These expressions show clearly that $(2, 1, 1, 1)$ can be decomposed to give a part in $\sigma_2$ and a part not in $\sigma_1$, whereas $(3, 1, 1, 1)$ can be decomposed to give a part in $\sigma_1$ and a part not in $\sigma_2$. Moreover, by linear independence of this set of four vectors, these are the only possible decompositions into sums of bundles in the affine spans of $\sigma_1, \sigma_2$. So neither is in $\sigma_1 + \sigma_2 = \sigma = \text{Conv } D_U(1, 5, 4, 2)$. But this means that the only integer vectors in $\text{Conv } D_U(1, 5, 4, 2)$ are in fact in $D_U(1, 5, 4, 2)$ itself: it is discrete-convex.

However, by changing which combinations of these four vectors pertain to each agent, we change the situation:

**Example A.30.** Now suppose Agent 1 has valuation $u^1(0, 0, 0, 0) = 0$, $u^1(1, 1, 0, 0) = 6$, $u^1(0, 1, 1, 0) = 9$ and Agent 2 has valuation $u^2(0, 0, 0, 0) = 0$, $u^1(0, 0, 1, 1) = 6$, $u^2(4, 0, 0, 1) = 6$. Now $\sigma^1 = \text{Conv}((0, 0, 0, 0), (1, 1, 0, 0), (0, 1, 1, 0))$ and $\sigma^2 = \text{Conv}((0, 0, 0, 0), (0, 0, 1, 1), (4, 0, 0, 1))$. The analysis proceeds very similarly to before, but in this case the decompositions (2) and (3) show clearly that both $(2, 1, 1, 1)$ and $(3, 1, 1, 1)$ are in $\sigma^1 + \sigma^2$. Thus they are in $\text{Conv } D_U(1, 5, 4, 2)$ but not in $D_U(1, 5, 4, 2)$: this set is in this case not discrete-convex.

In both Example A.29 and A.30 we can calculate the mixed volume relevant at the intersection: in both cases $MV_4(\sigma^1, \sigma^2, (2, 2)) = 3$. In both cases, the weights of the individual’s 2-cells that meet at $(1, 5, 4, 2)$ are both 1. There appears to be no way to distinguish Examples A.29 and A.30 using the tools developed here, and so we reiterate: the condition of Theorem 5.3 is sufficient, but not necessary, for existence of equilibrium when $n \geq 4$.

### A.4.3 The results of Bertrand and Bihan (2007, 2013)

Many authors have developed how one should think about tropical intersection multiplicities in order to translate results in classical algebraic geometry across to the tropical world (Sturmfels, 2002, Mikhalkin, 2005, Gathmann and Markwig, 2008, Osserman and Payne, 2013). In particular, a 2-dimensional tropical Bézout-Bernshtein theorem dates back to Sturmfels (2002) at the very beginning of the discipline of tropical geometry. In this paper, we appeal to the conventions and definitions of Bertrand and Bihan, who provided the general case in their 2007 preprint, the relevant parts of which are published in their (otherwise different) 2013 book chapter.
Bernshtein's (1975) theorem (see also Kouchnirenko, 1976) generalised Bézout’s (1779) by showing that the number of intersections of general curves in affine (complex) space is given by the mixed volume of their Newton polytopes (as opposed to assuming that the curves were of pure degree, and potentially counting many intersections ‘at infinity’). Recall that we defined the mixed volume in the text (Definition 5.1) as being a linear combination of volumes of sums of convex sets. We use this definition because it requires the least additional background in geometry. However, there are many equivalent definitions, and ours is not the most intuitive for those with more background in the subject. Cox et al (2005, Section 7.4) give much more explanation.\footnote{Note that there are two competing conventions for the mixed volume; some writers divide the form given here by \(n!\). We use the same convention as Cox et al (2005) and Bertrand and Bihan (2007, 2013).}

Also note that \(n\)-dimensional mixed volume takes \(n\) arguments and calculates the volume in dimension \(n\)–which is zero on any object which has fewer than \(n\) dimensions itself.

We define the weight of a general TH cell using the discussion of volumes of lattice polytopes (Appendix A.5.2).

**Definition A.31** (Bertrand and Bihan, 2013). The weight \(w_u\) of a \(k\)-cell \(C_{\sigma}\) of a TH, associated to the \((n-k)\)-cell \(\sigma\) of the SNP, is given by

\[
 w_u(C_{\sigma}) = (n-k)! \text{Vol}_{N_{\sigma}}(\sigma)
\]

We use again the notation introduced at the beginning of Appendix A.4. Now we can define the tropical intersection multiplicity:

**Definition A.32** (Bertrand and Bihan, 2013, Definition 5.2). Using the notation introduced at the beginning of Appendix A.4, the multiplicity of cell \(C\) in the intersection of THs \(\mathcal{T}_{u_1}, \ldots, \mathcal{T}_{u_r}\) is defined as follows:

1. If the intersection is transverse at \(p\) then
   \[
   \text{mult}(C) := [N_{\sigma} : N_{\sigma_1} + \cdots + N_{\sigma_r}] \cdot \prod_{j=1}^r w_{u_j}(C_{\sigma_j})
   \]
2. If the intersection is not transverse at \(p\), translate the THs by small generic vectors (as in Lemma 4.6) so that all intersections emerging from \(C\) are transverse. Define \(\text{mult}(C)\) as the sum of the multiplicities of the transverse intersections emerging from \(C\) which are cells of dimension \(n-d\).

It is useful to recall from Lemma A.25 that, in our notation, an intersection is transverse iff \(d^1 + \cdots + d^r = d\).

**Theorem A.33** (Bertrand and Bihan, 2013, Thm. 6.1). Use the notation of Appendix A.4.

1. If the intersection of \(\mathcal{T}_{u_1}, \ldots, \mathcal{T}_{u_r}\) is transverse at \(C\), then
   \[
   \text{mult}(C) = \text{MV}_d(\sigma^1, \ldots, \sigma^r; (d^1, \ldots, d^r)).
   \]
(2) In general, when \( d \leq d^1 + \cdots + d^r \), we have

\[
\text{mult}(C) = \sum_{t^1 + \cdots + t^r = d; \, \nu \geq 1} MV_d(\sigma^1, \ldots, \sigma^r; (t^1, \ldots, t^r)),
\]

where the sum is over all \( r \)-tuples \((t^1, \ldots, t^r)\) such that \( t^1 + \cdots + t^r = d \) and \( t^j \geq 1 \) for all \( j \). In particular, if \( d = r \), then \( \text{mult}(C) = MV_d(\sigma^1, \ldots, \sigma^r) \).

We state the following result in less generality than do Bertrand and Bihan, and in different language, so that its use for our purposes is clearer.

**Lemma A.34** (Bertrand and Bihan, 2013, Lemma 6.7). Suppose the intersection of \( \mathcal{T}_{u^1}, \mathcal{T}_{u^2} \) is transverse.\(^{\text{99}}\)

\[
MV_n(\tilde{A}^1, \tilde{A}^2, (d^1, d^2)) = \sum_{\dim \sigma^j = d^j} MV_n(\sigma^1, \sigma^2; (d^1, d^2))
\]

where the sum is taken over all cells \( \sigma = \sigma^1 + \sigma^2 \) of the SNP of \( U \), such that \( \dim \sigma^j = d^j \) for \( j = 1, 2 \), and such that \( \sigma^1, \sigma^2 \) correspond to TH cells which intersect along an aggregate TH cell corresponding to \( \sigma \).

### A.4.4 Proofs of results in Appendix 5.2

We can finally prove the results stated in the text.

**Lemma A.35.** Suppose \( \mathcal{T}_{u^1} \) and \( \mathcal{T}_{u^2} \) intersect transversally at the respective cells \( C^1, C^2 \). Then \( w_{u^1}(C^1)w_{u^2}(C^2) \leq MV_n(\sigma^1, \sigma^2; (d^1, d^2)) \), with equality holding iff \([N_{\sigma^1} : N_{\sigma^1} + N_{\sigma^2}] = 1\).

**Proof.** By Theorem A.33.1 we know that \( \text{mult}(C) = w_{u^1}(C^1)w_{u^2}(C^2)[N_{\sigma^1} : N_{\sigma^1} + N_{\sigma^2}] = MV_n(\sigma^1, \sigma^2; (d^1, d^2)) \), from which the result follows. \( \square \)

**Proof of Lemma 5.2.** We have THs \( \mathcal{T}_{u^1} \) and \( \mathcal{T}_{u^2} \). Let \( C \) be a 0-cell of their intersection, and use the notation of Appendix A.4. In particular, the \( d \) from that notation is equal to \( n \) and, since the intersection is transverse, we know \( n = d^1 + d^2 \) (recalling that \( d^j = \dim \sigma^j = n - \dim C^j \)).

Now we take the sum over all 0-cells of the intersection which are the intersection of a \((n - d^1)\)-cell of \( \mathcal{T}_{u^1} \) and an \((n - d^2)\)-cell of \( \mathcal{T}_{u^2} \), and apply Lemmas A.34 and A.35, to see that the sum of such 0-cells, weighted only by the products of weights of the intersecting cells, is bounded above by \( MV_n(\tilde{A}^1, \tilde{A}^2, (d^1, d^2)) \). Moreover, equality holds iff \([N_{\sigma} : N_{\sigma^1} + N_{\sigma^2}] = 1\) in every case. The result now follows by Proposition A.28. In particular, if \( n = 2 \) then every transverse intersection of cells is an intersection of two facets, and if \( n = 3 \) then every transverse intersection of cells is an intersection of two facets or a facet and a 1-cell. So in these cases there is at most one corresponding SNP cell with dimension \( >1 \), and hence (by Proposition A.28) \([N_{\sigma} : N_{\sigma^1} + N_{\sigma^2}] > 1\) demonstrates failure of equilibrium. \( \square \)

\(^{\text{99}}\)The assumption of transversality here is equivalent to the condition, in the language of Bertrand and Bihan, 2013, Lemma 6.7, that the ‘convex mixed subdivision’ is ‘pure’.

63
Proof of Theorem 5.3. That the upper bound holds is obvious from Lemma 5.2.

If the naively-weighted count is equal to the upper bound, then by Lemma 5.2, the demand set is discrete-convex for every 0-cell in the intersection of the THs. Now suppose that equilibrium fails for some supply. By Lemma 4.4, the demand set is not discrete-convex at some price in the intersection, and hence it is not discrete-convex for any price in the interior of the corresponding cell in the TH of aggregate demand. But by Corollary A.11, because we assume the domain of aggregate demand to be in $n$ dimensions, there is a 0-cell in the boundary of this cell, at which price the demand set is also not discrete-convex. Since a TH intersection is closed, this 0-cell is also in the intersection of the individual THs: we have a contradiction. So in this case, equilibrium exists for every supply.

If the naively-weighted count is strictly below the upper bound, and if $n \leq 3$, then it is immediate by Lemma 5.2 that equilibrium must fail for some supply. □

A.4.5 Proofs for Section 5.4: results when the intersection is not transverse

First, an interesting example of a non-transverse intersection, such that competitive equilibrium does exist for all supplies in the domain of aggregate demand, only if we do not make any small shift of the valuations.

Example A.36. Consider two identical agents whose demand sets at price $(2,2)$ are the bundles $(0,0), (1,2), (2,1)$ and $(1,1)$. (For example, this is consistent with a valuation $u(x,0) = x; u(0,y) = y; u(1,1) = 4; u(1,2) = u(2,1) = 6; u(2,2) = 7$). Then the bundle $(2,2)$ is in the aggregate demand set at this price: we assign bundle $(1,1)$ to both agents. But observe that bundle $(1,1)$ is an interior point of each agent’s SNP cell with the vertices $(0,0), (1,2)$, and $(2,1)$, so if we make a small perturbation to either agent’s valuation so their THs intersect transversally, then for any prices close to $(2,2)$ the perturbed agent’s demand must be some subset of these vertices, and it is easy to see that $(2,2)$ cannot be an aggregate demand.

We now state the full version of Maclagan and Sturmfels (2015) Proposition 3.6.12.

Proposition A.37 (Maclagan and Sturmfels, 2015, Proposition 3.6.12). For any THs $T_u^1$ and $T_u^2$, and generic $v \in \mathbb{R}^n$, the limit $\lim_{\epsilon \to 0} T_u^1 \cap (\epsilon v + T_u^2)$ exists and equals the stable intersection of $T_u^1$ and $T_u^2$.\footnote{The limit is taken in the Hausdorff metric.}

Additionally, the proof of the following is clear from the proof of Proposition 3.6.12 of Maclagan and Sturmfels, 2015.

Corollary A.38. For sufficiently small $\epsilon > 0$ the combinatorial type of $T_{U_{\epsilon}}$ is independent of $\epsilon$. □

This enables us to prove Theorem 5.5

Proof of Theorem 5.5. If such a price exists, then clearly equilibrium fails for this supply (see Lemma 2.9).

Conversely, suppose equilibrium does not exist for $x$ in the domain of aggregate valuation. Then, combining Proposition 4.6 with Lemmas A.18 and A.19, there exists $w$...
such that, for all $\epsilon > 0$ and beneath some upper bound, the THs $T_{u^1}$ and $\epsilon w + T_{u^2}$ meet transversally everywhere, and equilibrium fails for supply $x$ when the agents have the corresponding valuations. Fix suitable $\epsilon$, and write the corresponding aggregate valuation as $U_\epsilon$. Note by Corollary A.38 that the combinatorial type of $T_{U_\epsilon}$ is independent of $\epsilon$ for such $\epsilon$.

As $x$ is in the domain of the aggregate valuation, we have $x \in \sigma_\epsilon$ for some SNP cell $\sigma_\epsilon$ of the SNP corresponding to $U_\epsilon$. Moreover, we assumed that the domain of $U$ has dimension $n$, from which it follows that $\sigma_\epsilon$ has dimension $n$, and corresponds to a 0-cell of $T_{U_\epsilon}$. Let $p_\epsilon$ be the price at this 0-cell. Then $x \in \text{Conv} D_{U_\epsilon}(p_\epsilon)$. However, we know that $x \notin D_{U_\epsilon}(p)$ for any $p \in \mathbb{R}^n$. We conclude, as argued at Lemma 4.4, that $p_\epsilon$ is at the intersection of the individual tropical hypersurfaces, that is, $p_\epsilon \in T_{u^1} \cap (\epsilon w + T_{u^2})$.

Identify minimal cells $C^1_\epsilon$ of $T_{u^1}$ and $C^2_\epsilon$ of $\epsilon w + T_{u^2}(= T_{u^2})$ such that $p_\epsilon \in C^1_\epsilon \cap C^2_\epsilon$. Since $p_\epsilon$ is at a 0-cell of $U_\epsilon$, by minimality of $C^1_\epsilon$ and $C^2_\epsilon$, this intersection must be at only one point: $\{p_\epsilon\} = C^1_\epsilon \cap C^2_\epsilon$.

Now note that $C^2_\epsilon = \epsilon v + C^2$, where $C^2$ is a cell of $T_{u^2}$, by Lemma A.18. So $p_\epsilon \in C^1_\epsilon \cap (\epsilon v + C^2)$, and hence in particular the latter is non-empty. As we chose sufficiently small $\epsilon$ that the combinatorial type of $T_{U_\epsilon}$ is independent of $\epsilon$, it follows that $C^1_\epsilon \cap (\epsilon' v + C^2)$ contains one point, $p_{\epsilon'}$, for all $\epsilon' \in (0, \epsilon)$. Hence, since cells are closed, the limit as $\epsilon \to 0$ is a 0-cell of the stable intersection. Let $p$ be the price at this 0-cell. Obviously $p \in C^1_\epsilon \cap C^2_\epsilon$.

Now let $\sigma^1$ and $\sigma^2$ be the SNP cells of the individual valuations corresponding to $C^1_\epsilon$ and $C^2$. Of course $\sigma^2$ is also the SNP cell for agent 2′ corresponding to cell $C^2_\epsilon = \epsilon v + C^2$. So $\sigma_\epsilon = \sigma^1 + \sigma^2$. Since $p \in C^1_\epsilon \cap C^2$ we know $\sigma^1 \subseteq D_{u^1}(p)$ and $\sigma^2 \subseteq D_{u^2}(p)$. Hence $x \in \sigma_\epsilon = \sigma^1 + \sigma^2 \subseteq \text{Conv} D_{u^1}(p) + \text{Conv} D_{u^2}(p) = \text{Conv} D_{U}(p)$. \hfill $\square$

Our remaining theorem is also easy to prove from Bertrand and Bihan’s results and definitions.

**Proof of Theorem 5.6.** Take a small translation of $T_{u^2}$ so that the intersections are all transverse. By Definition A.31.2, the weight of any 0-cell in the stable intersection (before this translation) is given by the sum of the weights of the 0-cells that emerged from it (after this translation); by Proposition A.37, such 0-cells will indeed emerge. As these weights are all positive integers, then, the number of 0-cells in the stable intersection is bounded above by the weighted sum of 0-cells after the translation, which, by Lemma A.34, is the number stated. \hfill $\square$

### A.5 Mathematical background for the proofs in Appendix A.4: Lattices and subgroup indices

The following material is intended as a companion to Appendix A.4.

#### A.5.1 Lattices

The key to the question of equilibrium with indivisible goods, is that available bundles form a subset of a lattice, and individual agents’ demands aggregate as lattices. So, before considering the parallel lattice associated with an SNP cell, we cover some preliminary material on lattices. We will only be interested in lattices within $\mathbb{Z}^n$. Readers
familiar with group theory will recognise that lattice are defined here as is an additive subgroups of \( \mathbb{Z}^n \). Familiarity with group theory is not essential to understand this section, but may deepen understanding a little.

**Definition A.39.**

1. A **lattice** is a set \( N \subseteq \mathbb{Z}^n \) such that \( 0 \in N \) and if \( n, n' \in N \) then \( n - n' \in N \).
2. \( M \) is a sublattice of \( N \) if \( M \subseteq N \) and \( M \) has the structure of a lattice.
3. The **linear span** of a lattice is the minimal vector subspace of \( \mathbb{R}^n \) containing \( N \).
4. The **rank** of a lattice is the dimension of its linear span.
5. An **integer basis** for a lattice \( N \) is a set \( \{n^1, \ldots, n^r\} \) such that any \( n \in N \) can be uniquely presented as \( n = \sum_i \alpha_i n^i \) for \( \alpha_i \in \mathbb{Z} \).

We refer to integer bases rather than just bases for lattices to retain clarity that these are bases for lattices and not just the linear space they span. We will be particularly interested in sublattices of equal rank.

We now group some important results.

**Fact A.40.**

1. If \( N, M \subseteq \mathbb{Z}^n \) are lattices, then the Minkowski sum \( N + M \) is a lattice.
2. Any lattice has an integer basis.\(^{101}\)
3. If \( M, N \) are lattices whose linear spans have zero intersection and if \( \{m^1, \ldots, m^r\}, \{n^1, \ldots, n^s\} \) are integer bases for them respectively, then \( \{m^1, \ldots, m^r, n^1, \ldots, n^s\} \) is an integer basis for \( M + N \).
4. Suppose the rank \( k \) lattice \( N \) is equal to \( \mathbb{Z}^n \cap L_N \) where \( L_N \) is its linear span. A set \( \{n^1, \ldots, n^k\} \) of linearly independent vectors in \( N \) is a basis iff it is unimodular.

We emphasise in particular Fact A.40.4, an important result which was also mentioned in Remark A.15.

We can therefore define:

**Definition A.41.** A **fundamental parallelepiped** of a lattice \( N \) is the set \( \{ \sum_i \lambda_i n^i \mid 0 \leq \lambda_i \leq 1 \} \), where \( \{n^1, \ldots, n^r\} \) are a basis for \( N \).

Note that a different basis will give a different parallelepiped, but they will always be related by a unimodular basis change.

**A.5.2 Volumes of Lattice Polytopes, and subgroup indices**

**Definition A.42.** The **\( n \)-dimensional volume** of a convex set \( X \) in \( \mathbb{R}^n \) is:

\[
\text{Vol}_n(X) := \int \cdots \int_X 1 \, dp_1 \cdots dp_n.
\]

\(^{101}\)This is the ‘fundamental theorem on discrete subgroups of Euclidean spaces’, see Cox et al., 2005, p334.
If the dimension $k$ of $X$ is less than $n$ then this will always be zero. We need, however, some measure of the $k$-dimensional volume of a $k$-dimensional polytope which lives in $\mathbb{R}^n$.\footnote{One would usually determine the $k$-dimensional volume of a $k$-dimensional subset $X$ of $\mathbb{R}^n$ by using an orthonormal change of basis matrix so that $X \subseteq \mathbb{R}^k$ for some fixed subset of coordinates of $\mathbb{R}^n$. Then the volume as defined above can be taken.} Moreover, we wish to normalise so that, for example, an SNP edge comprising only one copy of a primitive integer vector has ‘length 1’. Similarly, will consider the fundamental parallelepiped of a rank $k$ sublattice $N \subseteq \mathbb{Z}^n$ to have $k$-dimensional volume equal to $1$ with respect to this lattice. Thus the appropriate change of basis is not orthogonal, but defined by a basis for the lattice in question.

Specifically, take such a basis $\{n^1, \ldots, n^n\}$ for $N$ and extend it to a basis for $\mathbb{R}^n$, for example by appending suitable coordinate vectors (the choice of these vectors will not be relevant).\footnote{Those familiar with abstract linear transformations will see that choosing such vectors is unnecessary; we include this step so readers unfamiliar with such material can think entirely in terms of square matrices.} Let $G_N$ be the inverse of the matrix with these vectors as its columns, so $G_N n^i = e^i$ for $i = 1, \ldots, k$. Then $G_N$ restricts to an isomorphism between $N$ and $\mathbb{Z}^k$: any polytope $X$ with vertices in $N$ is mapped under $G_N$ to a polytope in $\mathbb{R}^k \subseteq \mathbb{R}^n$ with vertices in $\mathbb{Z}^k \subseteq \mathbb{Z}^n$. With these conventions:

**Definition A.43.** The *lattice-volume* a polytope $X$ with vertices in a lattice $N \subseteq \mathbb{Z}^n$ is $\text{Vol}_N(X) := \text{Vol}_k(G_N X)$.\footnote{Those familiar with group theory will recognise that $[N : M]$ is the subgroup index in the ordinary sense: it is the number of cosets of $M$ in $N$, that is, the number of disjoint sets $n + M$ where $n \in N$. It is standard group theory that each such coset may be represented by some $n$ in the fundamental parallelepiped of $M$, and if $n + M \neq M$ then such $n$ is not a vertex of this parallelepiped, and is unique; these points both follow from the simple observation that $n + M = n' + M \Leftrightarrow n - n' \in M$. This shows part 1. In part 2, the fundamental point is that $[N : M]$ is the determinant of the $(k \times k)$ change of basis from $M$ to $N$, but again we present an explicit $n \times n$ matrix with the requisite property so readers need not concern themselves unnecessarily with unfamiliar material.}

It is easy to see that this is independent of the choice of basis for $N$ (and its extension to $\mathbb{R}^n$) and that the volume of the fundamental parallelepiped in $N$ is $1$.

Now the sublattice index easy to define.

**Definition A.44.** Let $M \subseteq N$ be a sublattice of equal rank, and let $\Delta_M$ be a fundamental parallelepiped of $M$. The *subgroup index* $[N : M]$ is $\text{Vol}_N(\Delta_M)$.

The following points are standard:\footnote{Those familiar with group theory will recognise that $[N : M]$ is the subgroup index in the ordinary sense: it is the number of cosets of $M$ in $N$, that is, the number of disjoint sets $n + M$ where $n \in N$. It is standard group theory that each such coset may be represented by some $n$ in the fundamental parallelepiped of $M$, and if $n + M \neq M$ then such $n$ is not a vertex of this parallelepiped, and is unique; these points both follow from the simple observation that $n + M = n' + M \Leftrightarrow n - n' \in M$. This shows part 1. In part 2, the fundamental point is that $[N : M]$ is the determinant of the $(k \times k)$ change of basis from $M$ to $N$, but again we present an explicit $n \times n$ matrix with the requisite property so readers need not concern themselves unnecessarily with unfamiliar material.}

**Fact A.45.** Using the notation defined above, and in particular using the *same* vectors in $\mathbb{R}^n$ to extend a basis for either $N$ or $M$ to $\mathbb{R}^n$, we have

1. $[N : M] - 1$ is equal to the number of non-vertex points of $N$ in $\Delta_M$.
2. $[N : M] = |\text{det}(G_N G_M^{-1})|$.

**A.6 Proofs for Section 6**

**Proposition A.46.** The demand type of Section 6.4 is unimodular, and is not a basis change from 4-dimensional strong substitutes $D^4_{ss}$. 
Proof. One may easily show this demand type is unimodular.\textsuperscript{105}

To show it is not a basis change from 4-D strong substitutes, \( \mathcal{D}_{ss}^4 \), assume (for contradiction) there exists a unimodular matrix \( G \) such that \( G^{-1}D \) consists entirely of distinct column vectors from \( \mathcal{D}_{ss}^4 \).\textsuperscript{106} Since \( D \) has 9 columns, \( G^{-1}D \) must include all but one of the 10 distinct vectors in \( \mathcal{D}_{ss}^4 \). Let \( w := (1, 1, 1, 1, 1, -1, -1, -1) \) and note that \( Dw' = 0 \), so \( G^{-1}Dw' = 0 \) also. It follows that every row \( r \) of \( G^{-1}D \) satisfies \( r.w = 0 \). But there are precisely four vectors in \( \mathcal{D}_{ss}^4 \) with non-zero entry in any coordinate \( i \) (\( e^i \), and \( e^i - e^j \) for the three values of \( j \neq i \)), so there are four non-zero entries in every row of the matrix whose columns are the 10 distinct vectors of \( \mathcal{D}_{ss}^4 \), and if we delete any one column, then at least one row must have exactly three non-zero entries. Since these three entries are \( \pm 1 \), there is no way to add or subtract the three together to obtain zero; it is impossible that this row has zero dot product with \( w \). Thus no nine vectors of \( \mathcal{D}_{ss}^4 \) can form the columns of \( G^{-1}D \), for any unimodular matrix \( G \). \( \square \)

As noted in the text, our matrix \( D \) does not contain all the coordinate vectors. There are many basis changes that do, for example:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\textsuperscript{105}For example, it can be confirmed using Matlab that the determinant of every set of four columns of \( D \) is \( \pm 1 \) or 0.

\textsuperscript{106}Vectors which are the negation of one another are not considered “distinct” in this context.