

Signaling in Matching Markets

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Abstract

We evaluate the effect of costless preference signaling in two-sided matching markets between firms and workers. We consider a game of incomplete information with firm segments. Workers agree on the ranking of firms across segments, but have idiosyncratic and uniformly distributed preferences within segments. Firm preferences over workers are idiosyncratic and uniformly distributed. Each worker can send a limited number of signals to firms. Then, each firm makes an offer to a worker. Finally, workers choose an offer from those available to them. We show that, on average, introducing a signaling mechanism increases both the expected number of matches as well as the expected welfare of workers for this environment. The welfare of firms, on the other hand, changes ambiguously. In addition, the signaling mechanism adds the most value for markets wherein the number of firms and the number of workers are of roughly the same magnitude. Furthermore, the optimal number of signals—the number of signals that maximizes the expected increase in the number of matches—increases in the number of worker positions. Finally, additional periods of interaction between firms and workers decrease the impact of signaling.

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1 Introduction

Labor markets often have the feature that job seekers apply for many positions, as the low cost of application relative to the high value of being employed induces job seekers to perform broad searches. Employers, consequently, face the task of evaluating numerous applications and deciding which candidates to pursue. Much of the evaluation process entails gathering information about the candidates' adequacy for any given position. However, pursuing candidates is a costly process, and each employer must carefully assess whether the candidates—many of whom may be performing broad job searches—would in fact be likely to accept its offer.

At the same time, candidates who have preferences for a particular employer may have an incentive to convey this information, a behavior we term *preference signaling*, since this may increase the likelihood of ultimately receiving a job offer. For preference signaling to be effective, employers must take the information seriously. But preference signaling often bears almost no cost, and job seekers have an incentive to indicate a particular interest to many employers, regardless of how strong their preferences towards these employers actually are. Employers, therefore, must further discern which preference information is sincere and which is simply cheap talk. So while employers may value learning candidate preferences, and candidates may wish to signal their preferences, inability to credibly convey information may prevent any gains from preference signaling from being realized.

In this paper, we investigate how a *signaling mechanism* that limits the number of signals a job seeker sends can overcome the credibility problem and improve the welfare of labor market participants. In our model, firms make offers to workers, but they are uncertain about worker preferences. Worker preferences are correlated in that the workers agree on a ranking of blocks of firms, but have idiosyncratic and uniformly distributed preferences within each block. Firm preferences are idiosyncratic and uniformly distributed. Workers have the opportunity to send identical signals to a limited number of firms before the firms make offers. Firms observe their signals (but do not observe the signals to other firms) and make offers to workers simultaneously. Finally, workers choose offers from those available to them.

We show that, on average, introducing a signaling mechanism increases both the expected number of matches as well as the expected welfare of workers. The welfare of firms, on the other hand, changes ambiguously. Intuitively, when firms make offers to workers who send them signals, these offers are less likely to overlap, which increases the expected number of matches. Furthermore, since in equilibrium workers send signals to their most-preferred firms within blocks and firms respond to signals, i.e., make offers to signaling workers, workers on average receive better offers than without signals, so that expected worker welfare increases with the addition of a signaling mechanism. On the other hand, when a firm makes an offer

to a worker who has signaled it, this creates strong competition for firms who would like to make the same worker an offer, but have not received a signal. Hence, by responding to signals firms can generate a negative spillover on other firms.

We also compare *the value of a signaling mechanism*—the expected increase in the number of matches from the introduction of a signaling mechanism—across markets where agents care about getting a match, but not the quality of the match. For such an environment, we show that the value of the signaling mechanism is maximal for balanced markets; that is, markets wherein the number of firms and the number of workers are of roughly the same magnitude. Furthermore, additional periods of interaction between firms and workers decrease the impact of signaling. Lastly, we analyze markets in which workers can send several signals and can “occupy” several positions (for example, when positions correspond to interviews). We show that the optimal number of signals—the number of signals that maximizes the value of a signaling mechanism—increases when workers have more positions to fill.

Preference signaling is manifested in numerous labor markets, and in some markets, variants of the signaling mechanism we analyze here have recently emerged. Since 2006 the American Economic Association (AEA) has operated a service to facilitate the job search for economics graduate students nearing completion of their degrees. Through this service, students can send signals to up to two employers to indicate their interest in receiving an interview at the Allied Social Science Associations meeting, an annual conference at which departmental recruiting committees conduct interviews with candidates for open faculty positions.¹ Skydeck360, a student-operated company at Harvard, offers a similar signaling service for MBA students in their search for internships and full-time jobs. Each registered student can send up to ten signals to employers via their secure website. Unlike the AEA’s signaling mechanism, Skydeck360 offers tiers of signals: students designate three signals to their “top choice” employers, differentiating these from the remaining signals.

Similarly, some online dating websites also employ signaling mechanisms, wherein agents send signals to potential partners to indicate special interest. For example, in the dating service Cupid.com members have a limited number of “virtual roses” they may attach to messages. Since women using dating websites are often flooded with messages from men, the scarce roses provide men with a means of indicating their interest is genuine, and that they have not simply made contact by thoughtlessly casting a wide net. Lee et al. (2009) run a field experiment with the major Korean online dating website Couple.net to measure potential gains from introducing a signaling mechanism similar to that of Cupid.com. They find that users of both genders are more likely to accept a dating request when a “virtual rose” is attached, and that roses are most helpful in improving acceptance rates for senders

¹For details, see “Signaling for Interviews in the Economics Job Market” on the American Economic Association website: <http://www.aeaweb.org/joe/signal>.

of “below average popularity.”

The signaling mechanism in our paper is closely related to the strategic information transmission between a sender and a receiver in the seminal paper of Crawford and Sobel (1982). In our model, however, we consider a multi-stage game with many senders (workers) and many receivers (firms), where the structure of allowable signals plays a distinctive role. Senders must choose the receivers to whom they will send one of their limited identical signals, and the scarcity of signals induces credibility. Each receiver knows only whether a sender has sent a signal to it or not, and receives no additional information. Nevertheless, some features of Crawford and Sobel (1982) persist in our model. Signals are “cheap” in the sense that they do not have a direct influence on agent payoffs. Each agent has only a limited number of signals; so there is an opportunity cost associated with sending a signal. Lastly, in our model there always exist babbling equilibria where agents ignore signals; hence, the introduction of a signaling mechanism always enlarges the set of equilibria.

While in this paper we find that signaling mechanisms can improve agent welfare under a broad class of preferences, for some agent preferences, signaling can serve to worsen outcomes. Kushnir (2009) analyzes the mechanism introduced in this paper when worker preferences are “almost aligned;” that is, when each worker has “typical” preferences with probability close to one and idiosyncratic preferences with complementary probability close to zero. Firms have commonly known preferences over workers. In the absence of signals, firms make non-overlapping offers to workers, resulting in the maximum number of possible matches. Signals disturb firms’ commonly held beliefs about workers preferences, and this asymmetry in turn disrupts the maximal matching. While we have modeled a general, low-information setting of market congestion, Kushnir (2009) serves as a reminder that signaling mechanisms are not appropriate for all settings. At the very least, in high-information settings with minimal congestion, signals may do more harm than good.

We want to point out also that signaling in our paper differs fundamentally from the “job market signaling” first introduced in Spence (1973). Both in our model and in the examples discussed above, signals have no direct cost, and do not they convey information about worker ability. Rather, signals serve to credibly convey candidate preference information and to guide employer offers. Thus, they reduce coordination failure in congested labor markets.

Early college admission in the U.S. can be viewed as a form of preference signaling.² There are two predominant types of early admissions in the U.S.: early action programs, through which students may apply early but without any commitment to enroll, and early decision programs, through which students commit to enroll if accepted. Avery and Levin

²More than a hundred colleges adopted some form of early admission program in the 1990s Avery et al. (2003), and many schools fill a significant fraction of their entering class with early applicants.

(2009) consider a setting where an early application serves as a signal of enthusiasm about attending a particular college, and colleges derive utility from enthusiastic students. They show that selective (or elite) schools benefit from adopting an early action policy. At the same time, a lower ranked school, by adopting an early decision policy, can attract highly qualified but cautious students, drawing them away from more highly ranked schools.

The paper proceeds as follows. Section 2 introduces the offer game with and without signals. Equilibrium analysis is presented in Section 3. Section 4 considers the impact of a signaling mechanism on the welfare of agents. Additional welfare implications in a particular low-information setting are discussed in Section 5. Section 6 analyzes the robustness of the welfare results across markets with various sizes, numbers of signals, and periods of agents' interaction by considering a model in which agents care primarily about obtaining a match. Section 7 presents the conclusion.

2 Model

In this paper, a *market* consists of a set of firms, a set of workers, and a distribution over firm and worker preferences. We examine behavior in the market in two congested settings. In the first setting, which we term the *offer game with no signals*, each firm makes an offer to a worker based on a limited knowledge of worker preferences. In the second setting, *the offer game with signals*, before offers are made, each worker has the opportunity to send a costless signal to a firm, which in turn may use this signal to infer worker preferences.

Let $\mathcal{F} = \{f_1, \dots, f_F\}$ be the set of firms, and $\mathcal{W} = \{w_1, \dots, w_W\}$ be the set of workers, with $|\mathcal{F}| = F$ and $|\mathcal{W}| = W$. We consider markets with at least two firms and two workers. Firms and workers have preferences over each other as follows. Let Θ_f be the set of all possible firm preference lists (or rank order lists) over the workers, and let Θ_w be the set of all possible worker preference lists over the firms. Lists $\theta_f \in \Theta_f$ and $\theta_w \in \Theta_w$ are vectors of length W and F respectively. We sometimes refer to an agent that has rank r in agent a 's preference list as θ_a^r . Define $\Theta_F = (\Theta_f)^F$ and $\Theta_W = (\Theta_w)^W$, and let $\Theta \equiv (\Theta_f)^F \times (\Theta_w)^W$ indicate the set of all agent preference profiles. Let $t(\cdot)$ be the distribution over preference list profiles.

Each firm has the capacity to hire at most one worker, and each worker can fill at most one position.³ Firm f with preferences θ_f values a match with worker w as $u(\theta_f, w)$, where $u(\theta_f, \cdot)$ is a von-Neumann Morgenstern utility function. We assume that firm utility for a match depends only on a worker's rank. That is, for any permutation σ of worker indices, we have $u(\sigma(\theta_f), \sigma(w)) = u(\theta_f, w)$.⁴ We will sometimes abuse notation and write $u(\theta_f, \theta_f^j)$

³In Section 6 we also consider a setup in which each worker has more than one position to fill and can send more than one signal.

⁴Let $\sigma : \{1, \dots, W\} \rightarrow \{1, \dots, W\}$ be a permutation. Abusing notation, we apply σ to preference lists,

as $u(j)$, which we call firm utility from matching with its j th-ranked worker. Worker w with preference θ_w values a match with firm f as $v(\theta_w, f)$, where match utility again depends only on rank. Though not essential for our results, we will assume that workers and firms derive zero utility from being unmatched, and that any match is preferable to remaining unmatched for all participants.

Definition 1 A market is given by the 5-tuple $\langle \mathcal{F}, \mathcal{W}, t, u, v \rangle$.

We consider a special type of market: the *block-correlated market*.

Definition 2 A *block-correlated market* is a market $\langle \mathcal{F}, \mathcal{W}, t, u, v \rangle$ such that for a partition $\mathcal{F}_1, \dots, \mathcal{F}_B$ of the firms into blocks, ordinal preferences (as encompassed in $t(\cdot)$) are such that

1. For any $b < b'$, where $b, b' \in \{1, \dots, B\}$, each worker prefers every firm in block \mathcal{F}_b to any firm in block $\mathcal{F}_{b'}$;
2. Each worker's preferences within each block \mathcal{F}_b are uniform and uncorrelated; and
3. Each firm's preferences over workers are uniform and uncorrelated.

We call distributions $t(\cdot)$ that satisfy the criteria in Definition 2 *block uniform*. Block-correlated markets are meant to capture the notion that there are market segments in many two-sided markets. Agents largely agree on the ranking of agents on the other side across segments, but have idiosyncratic preferences within segments of the market. For example, workers might agree on the set of firms that constitute the “top tier” of the market; however within that tier, preferences are influenced by factors specific to each worker. We analyze only *block-correlated* markets.

2.1 The Offer Game with No Signals

We first examine behavior in a congested market in the absence of a signaling mechanism. Play proceeds as follows. First, preferences are realized for firms and workers. Next, each firm makes an offer to at most one worker; offers are made simultaneously. Finally, workers choose at most one offer from those available to them. Sequential rationality ensures that workers will always select the best offer from those available to them. Hence, we assume this behavior for the workers and focus on the reduced game with only firms as strategic players.

workers, and sets of workers, such that the permutation applies to the worker indices. For example, suppose $W = 3$, $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 1$. Then we have $\theta_f = (w_1, w_2, w_3) \Rightarrow \sigma(\theta_f) = (w_2, w_3, w_1)$ and $\sigma(w_1) = w_2$.

To analyze behavior and outcomes, we model this setting as a Bayesian game. Firm f type is simply its preferences θ_f , drawn from a block uniform distribution. Upon realization of its preference list, a firm must select a worker, if any, to whom to make an offer. A mixed strategy for firm f is a map from the set of preferences lists to the set of distributions over the union of workers with the no-offer option, denoted by \mathcal{N} , $s_f : \Theta_f \rightarrow \Delta(\mathcal{W} \cup \mathcal{N})$.⁵ We denote a profile of all firms' strategies as $s_F = (s_{f_1}, \dots, s_{f_F})$.

Let us denote function $\pi_f : (\Delta(\mathcal{W} \cup \mathcal{N}))^F \times \Theta \rightarrow \mathbf{R}$ as the payoff to firm f as a function of firms' strategies and realized agents' types. We are now ready to define the Bayesian Nash equilibrium of the offer game with no signals.

Definition 3 *Strategy profile \hat{s}_F is a Bayesian Nash equilibrium in the offer game with no signals, if for all $f \in \mathcal{F}$ and $\bar{\theta}_f \in \Theta_f$ we have*

$$\hat{s}_f(\bar{\theta}_f) \in \arg \max_{s_f \in \Delta(\mathcal{W} \cup \mathcal{N})} \mathbb{E}_{\theta_{-f}}(\pi_f(s_f, \hat{s}_{-f}, \theta) \mid \theta_f = \bar{\theta}_f)$$

We focus on equilibria when firm strategies depend only on workers' ranks within a firm's preference list (ruling out strategies that rely on worker index):

Definition 4 *Firm f 's strategy s_f is anonymous if for any permutation $\sigma \in \Sigma$, and for any preference profile $\theta_f \in \Theta_f$, we have $s_f(\sigma(\theta_f)) = \sigma(s_f(\theta_f))$.*

As an example, "always make an offer to my second-ranked worker" is an anonymous strategy, whereas "always make an offer to w_2 " is not. Due to the uniform distribution of firm preferences, there is a unique equilibrium when firms use anonymous strategies, where each firm optimally makes an offer to the highest-ranked worker on its preference list.

Theorem 1 *The unique equilibrium of the offer game with no signals when firms use anonymous strategies is $s_f(\theta_f) = \theta_f^1$ for all $f \in \mathcal{F}$ and $\theta_f \in \Theta_f$.*

Note that Theorem 1 requires that firms' strategies be anonymous only on the equilibrium path. Firm deviations that do not satisfy the anonymity assumption are still allowed.

2.2 The Offer Game with Signals

We now modify the game so that each worker may send a *signal* to exactly one firm. The game now proceeds in three stages:

1. Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.

⁵In other words, firms select elements of a W -dimensional simplex; $s_f(\theta_f) \in \Delta^W$, where $\Delta^W = \{x \in \mathbf{R}^{W+1} : \sum_{i=1}^{W+1} x_i = 1, \text{ and } x_i \geq 0 \text{ for each } i\}$.

2. Each firm makes an offer to at most one worker; offers are made simultaneously.
3. Each worker may accept at most one offer from the set of offers she receives.

Once again, sequential rationality ensures that workers will always select the best offer from those available to them. Hence, we assume this behavior for workers and focus on the reduced game consisting of the first two stages.

In the first stage, a worker sends a signal to a firm, or else chooses not to send a signal. A mixed strategy for worker w is then a map from the set of all possible preference lists to the set of distributions over the union of firms and the no-signal option, denoted by \mathcal{N} , $s_w : \Theta_w \rightarrow \Delta(\mathcal{F} \cup \mathcal{N})$.

In the second stage, each firm decides to which worker, if any, it will make an offer based on the firm's preference profile and the set of signals it receives in the first stage. A mixed strategy of firm f is then a map from the set of all possible preference lists and the set of all possible profiles of signals to the set of distributions over the union of workers and the no-offer option. That is, $s_f : \Theta_f \times 2^{\mathcal{W}} \rightarrow \Delta(\mathcal{W} \cup \mathcal{N})$, where $2^{\mathcal{W}}$ is the set of all subsets of workers. We denote a profile of all worker and firm strategies as $s_W = (s_{w_1}, \dots, s_{w_W})$ and $s_F = (s_{f_1}, \dots, s_{f_F})$ correspondingly.

The payoff to firm f is now a function of firms' and workers' strategies and realized agents' types, which we again denote as $\pi_f : (\Delta(\mathcal{F} \cup \mathcal{N}))^W \times (\Delta(\mathcal{W} \cup \mathcal{N}))^F \times \Theta \rightarrow \mathbf{R}$. As the offer game with signals is a multi-stage game of incomplete information, we consider sequential equilibrium as a solution concept.

Definition 5 *Strategy profile (\hat{s}_W, \hat{s}_F) and posterior beliefs $\hat{\mu}_f(\cdot|\bar{h})$ for each firm f and each subset of workers $\bar{h} \subset \mathcal{W}$ are a sequential equilibrium if*

- for any $w \in \mathcal{W}$, $\bar{\theta}_w \in \Theta_w$: $\hat{s}_w(\bar{\theta}_w) \in \arg \max_{s_w \in \Delta(\mathcal{F} \cup \mathcal{N})} \mathbb{E}_{\theta_{-w}}(\pi_w(s_w, \hat{s}_{-w}, \theta) \mid \theta_w = \bar{\theta}_w)$,
- for any $f \in \mathcal{F}$, $\bar{\theta}_f \in \Theta_f$, $\bar{h} \subset \mathcal{W}$:
 $\hat{s}_f(\bar{\theta}_f, \bar{h}) \in \arg \max_{s_f \in \Delta(\mathcal{W} \cup \mathcal{N})} \mathbb{E}_{\theta_{-f}}(\pi_f(s_f, \hat{s}_{-f}, \theta) \mid \theta_f = \bar{\theta}_f, h_f = \bar{h}, \mu_f = \hat{\mu}_f)$,

where \hat{s}_{-a} denotes the strategies of all agents except a , and beliefs are defined using Bayes' rule.⁶

We again focus on equilibria wherein agents use anonymous strategies. Note that in the definition below, for workers we only consider permutations Σ_B that permute firm ordering within blocks.

Definition 6 *Firm f 's strategy s_f is anonymous if for any permutation $\sigma \in \Sigma$, preference profile $\theta_f \in \Theta_f$, and subset of workers $h \subset \mathcal{W}$, we have $s(\sigma(\theta_f), \sigma(h)) = \sigma(s(\theta_f, h))$. Worker w 's strategy s_w is anonymous if for any permutation $\sigma \in \Sigma_B$ and preference profile $\theta_w \in \Theta_w$, we have $s_w(\sigma(\theta_w)) = \sigma(s_w(\theta_w))$.*

3 Equilibrium Analysis

Theorem 1 established the unique equilibrium of the offer game with no signals when firms use anonymous strategies, in which firms straightforwardly make offers to their most preferred workers. From this point on, we consider the offer game with signals, wherein firm and worker strategies are more subtle. Each firm that receives signals faces a fundamental tradeoff: it can make an offer to a higher-ranked worker who has not sent a signal to it, or else to a lower-ranked worker who has sent a signal, and who is potentially more likely to accept the offer.

We are interested in equilibria where firms within each block play symmetric anonymous strategies. That is, if firm f and firm f' belong to the same block $\mathcal{F}_b, b \in \{1, \dots, B\}$, they play the same anonymous strategies and have the same beliefs. We call such firm strategies and firm beliefs *block-symmetric*. We denote equilibria where firm strategies and firm beliefs are block-symmetric and worker strategies are anonymous and symmetric as *block-symmetric equilibria*.

Let us consider any block-symmetric sequential equilibrium. Since firms within each block \mathcal{F}_b use the same anonymous strategies, we can denote the ex-ante probability of a

⁶As usual in a sequential equilibrium, permissible off-equilibrium beliefs are defined by considering the limits of completely mixed strategies.

worker w receiving an offer from a firm within block \mathcal{F}_b , conditional on w sending a signal to it, as q^b . Similarly, let p^b denote the probability of w receiving an offer from a firm within block \mathcal{F}_b , conditional on w not sending a signal to it. We also denote the probability that a worker sends her signal to a firm within block \mathcal{F}_b as α_b , where $\alpha_b \in [0, 1]$ and $\sum_{b=1}^B \alpha_b \leq 1$.

The following proposition characterizes all block-symmetric sequential equilibria that satisfy criterion $D1$ of Cho and Kreps (1987).⁷

Proposition 1 (Worker Strategies) *Consider a block-symmetric sequential equilibrium that satisfies criterion $D1$. Then either*

1. for any $b \in \{1, \dots, B\}$, $q^b = p^b$, or
2. there exists $b_0 \in \{1, \dots, B\}$ such that $q^{b_0} > p^{b_0}$, and
 - for any $b \in \{1, \dots, B\}$ such that $\alpha_b > 0$, we have $q^b > p^b$, and if a worker sends her signal to block \mathcal{F}_b , she sends her signal to her most preferred firm within \mathcal{F}_b , and
 - for any $b' \in \{1, \dots, B\}$ such that $\alpha_{b'} = 0$, workers' strategies are optimal for any off-equilibrium beliefs of firms from block $\mathcal{F}^{b'}$.

Proposition 1 proves that there are two types of block-symmetric equilibria that satisfy criterion $D1$. Equilibria of the first type are babbling, where firms ignore signals. The outcomes of these equilibria coincide with the outcome in the offer games with no signals. Consequently, the signaling mechanism adds no value in this case. In equilibria of the second type, workers send signals only to their most preferred firm in each block, possibly mixing across these top firms. We call such worker strategies *top-block strategies*. Moreover, if in equilibrium worker w is not prescribed to signal to some block $\mathcal{F}^{b'}$, then worker w strategy, $\alpha_{b'} = 0$, should be optimal even if signals are interpreted by firms from block $\mathcal{F}^{b'}$ in the most favorable way for worker w (only such off-equilibrium beliefs survive criterion $D1$); i.e., upon receiving a signal from worker w each firm f in $\mathcal{F}^{b'}$ believes that it is w 's most preferred firm within block $\mathcal{F}^{b'}$:

$$\mu_f(\{\theta_w \in \Theta_w : f = \max_{\theta_w}(f' \in \mathcal{F}^{b'})\} | w \subset h_f) = 1.$$

We call such off-equilibrium beliefs *top-block beliefs* (note that firms on-equilibrium beliefs are also top-block).

⁷See proof of Proposition 1 in Appendix A for the definition of criterion $D1$ of Cho and Kreps (1987). Criterion $D1$ could be also replaced by "never a weak best response" of Cho and Kreps (1987) and "universal divinity" of Banks and Sobel (1987) without making a change to the statement of Proposition 1.

We now examine firm behavior, taking workers' strategies and firms' beliefs as fixed. We will assume that workers use symmetric top-block strategies and that firms have top-block beliefs⁸.

Consider a firm f that has received signals from the set of workers $h \subset \mathcal{W}$. Call f 's most preferred worker TRW_f (top-ranked worker) and f 's most preferred worker among h that have signaled it TSW_f (top-signal worker). Since these depend on f 's preferences and the workers who have signaled to firm f , we sometimes write $TRW_f(\theta_f)$ and $TSW_f(\theta_f, h)$.

Proposition 2 shows that firm f 's tradeoff is reduced to a binary decision between a worker whom the firm ranks highly (TRW_f) and a lower-ranked worker who has signaled to it (TSW_f), provided firms $-f$ use anonymous strategies and workers use symmetric top-block strategies.

Proposition 2 *Suppose firms $-f$ use anonymous strategies and workers use symmetric top-block strategies. Consider a firm f that receives signals from workers $h \subset W$. Then the expected payoff to f from making an offer to TSW_f is strictly greater than the payoff from making an offer to any other worker in h . The expected payoff to firm f from making an offer to TRW_f is strictly greater than the payoff from making an offer to any other worker from set $W \setminus h$.*

The symmetry of worker strategies and the anonymity of firm strategies dictate that for any two workers $w, w' \in h$, f 's expectation that these workers will accept an offer is identical. Hence, if f makes an offer to a worker in h , it should make that offer to its most preferred worker. The same logic holds for any two workers in $W \setminus h$. In order to understand the behavior of firms, we define a special type of strategy, the *cutoff strategy*.

Definition 7 (Cutoff Strategy) *Strategy s_f is a cutoff strategy for firm f if $\exists j_1, \dots, j_W \in \{1, \dots, W\}$, such that for any $\theta_f \in \Theta_f$ and any $h \subset W$,*

$$s(\theta_f, h) = \begin{cases} TSW_f & \text{if } \text{rank}_{\theta_f}(TSW_f) \leq j_{|h|} \\ TRW_f & \text{otherwise.} \end{cases}$$

where $|h|$ denotes the power of set h . We call (j_1, \dots, j_W) f 's cutoff vector, which has as its components cutoffs for each positive number of signals received.

Firm f , which employs a cutoff strategy, needs only look at the rank of the highest-ranked worker who signaled to it, conditional on the number of signals it has received. If the rank

⁸Firms on-equilibrium beliefs should be top-block in any block-symmetric equilibrium. In addition, a block-symmetric equilibrium satisfies criterion D1, if and only if worker strategies are optimal when firms' off-equilibrium beliefs are top-block beliefs. See the proof of Proposition 1 in Appendix A for details.

of this worker is above a certain cutoff, then the firm makes an offer to TSW_f . Otherwise the firm makes an offer to TRW_f . However, cutoffs depend on the number of signals the firm receives. This is because worker signals provide information about the signals the other firms receive, affecting the other firms' behavior, and hence the optimal decision for firm f . Note that a cutoff strategy is an anonymous strategy.

While we have defined cutoffs as integers, we can extend the definition to include all real numbers in the range $(1, W)$ by letting cutoff $j + \lambda$, where $\lambda \in (0, 1)$, correspond to mixing between cutoff j and cutoff $j + 1$ with probabilities $1 - \lambda$ and λ respectively. Note that this is equivalent to f making offers to TSW_f with a rank higher than j , randomizing between TRW_f and TSW_f when TSW_f has the rank of exactly j , and making offers to TRW_f otherwise.

The next proposition states that provided firms $-f$ use anonymous strategies and workers use symmetric top-block strategies, it is optimal for a firm to play cutoff strategies.

Proposition 3 (Optimality of Cutoff Strategies) *Suppose workers use symmetric, top-block strategies. Then for any strategy s_f for firm f , there exists a cutoff strategy that provides f with a weakly higher expected payoff than s_f for any anonymous strategies s_{-f} of opponent firms $-f$.*

Under the assumptions of the proposition, the probability that firm f 's offer to TRW_f will be accepted depends only on the number of signals firm f receives, and not on the identity of the signaled workers. In addition, the probability that firm f 's offer to TSW_f will be accepted has a probability equal to one. Hence, if f considers that making an offer to TSW_f is optimal, it will certainly make an offer to TSW_f with a higher rank, provided the number of signals it receives is the same.

The last result of the section establishes the existence of equilibria in the offer game with signals. We first demonstrate equilibrium existence requiring firms to use only cutoff strategies, and then use Proposition 3 to show that this step is not restrictive.

Theorem 2 *There exists a block-symmetric equilibrium where 1) workers play symmetric top-block strategies, and 2) firms play block-symmetric, cutoff strategies.*

4 Welfare

In this section we focus on theoretical results regarding the welfare of the agents. We refine the analysis of this section to the domain of cutoff strategies. Since, cutoff strategies could be represented by cutoff vectors, we can impose a natural partial order on them. Firm f 's cutoff strategy s'_f is greater than cutoff strategy s_f if all cutoffs of s'_f are weakly greater than

all cutoffs of s_f and at least one of them is strictly greater. We also call firm f responds more to signals than firm f' does, if s_f is greater than $s_{f'}$.

We now examine how a firm should adjust its behavior in response to changes in the behavior of opponents. We find that responding to signals is a case of strategic complements.

Proposition 4 (Strategic Complements) *Suppose workers play symmetric top-block strategies, and firms $-f$ use cutoff strategies. If firm $f' \in -f$ increases its cutoffs (responds more to signals), firm f will also optimally weakly increase its cutoffs.*

When opponent firms $-f$ make offers to workers who have signaled to them, it is risky for firm f to make an offer to a worker who has not signaled to it. Such worker has signaled to another firm (in firm f 's block or in another block) that is very much inclined to make her an offer. The greater this inclination on the part of the firm's opponents $-f$, the riskier it is for firm f to make an offer to TRW_f . Hence, the more inclined firm f is to make an offer to TSW_f as well.

Denote function $m : (\Delta(\mathcal{F} \cup \mathcal{N}))^W \times (\Delta(\mathcal{W} \cup \mathcal{N}))^F \times \Theta \rightarrow \mathbf{R}$ that yields the expected total number of matches in the market as a function of agent strategies and types. The next two propositions show the welfare impact of a single firm adjusting its strategy to respond more to signals.

Proposition 5 (Number of Matches) *Assume firms use cutoff strategies and workers use top-block strategies. Fix the strategies of firms $-f$ as s_{-f} . Let firm f 's strategy s_f differs from s'_f only in that s'_f has greater cutoffs (responds more to signals). Then, we have $\mathbb{E}_\theta[m(s'_f, s_{-f}, \theta)] \geq \mathbb{E}_\theta[m(s_f, s_{-f}, \theta)]$.*

To summarize the intuition of the proof, observe that when a firm f switches its offer from TRW_f to TSW_f , it is the other offers received by these workers that determine the impact on the total number of matches. If both workers have other offers, or if neither has another offer, the number of matches is unaffected. When exactly one worker has another offer, it is more likely to be TRW_f , as this worker has signaled to another firm, while TSW_f has not. Hence, making an offer to TSW_f leads to a greater expected total number of matches.

In addition to the increase in the expected number of matches, response to signals also unambiguously increases worker welfare.

Proposition 6 (Worker Welfare) *Assume firms use cutoff strategies and workers use top-block strategies. Fix the strategies of firms $-f$ as s_{-f} . Let firm f 's strategy s_f differs from s'_f only in that s'_f has greater cutoffs (responds more to signals). Then for each worker $w \in W$, $\mathbb{E}_\theta[\pi_w(s'_f, s_{-f}, \theta)] \geq \mathbb{E}_\theta[\pi_w(s_f, s_{-f}, \theta)]$.*

As a corollary of the above propositions, we can compare the welfare of workers and the expected number of matches in the unique equilibrium of the offer game with no signals and any non-babbling block-symmetric equilibrium of the offer game with signals. However, in order to have a strict comparison there should exist a block with at least two firms where workers send their signals with positive probability in the equilibrium of the offer game with signals. By contrast, firm welfare across these equilibria cannot be compared. As Example A1 in Appendix A illustrates, firm welfare may be higher with or without a signaling mechanism.

Theorem 3 *Let us consider a non-babbling block-symmetric equilibrium of the offer game with signals so that there is a block \mathcal{F}^b with at least two firms, such that $\alpha_b > 0$. Then, the expected number of matches and the expected welfare of workers in this equilibrium are strictly greater than the corresponding parameters in the unique equilibrium of the offer game with no signals. The welfare of firms in these equilibria is incomparable, i.e. the welfare of firms in a block-symmetric equilibrium of the offer game with signals could be greater or smaller than in the unique equilibrium of the offer game with no signals.*

The next section provides additional welfare results for markets with a single block of firms.

5 Equilibrium Ranking in a Single Block

In this section, we analyze the properties of markets in which there is a single block of firms; that is, worker and firm preferences over each other are uniform and uncorrelated. This simplification is useful because as a corollary of Proposition 1, in any non-babbling symmetric equilibrium, workers always send signals to their most preferred firms (as opposed to the multi-block case in which workers may adjust their weights over top firms). This allows us to compare equilibria in terms of firm and worker welfare.

By Proposition 3, if workers signal to their most preferred firms and the strategies of firms $-f$ are anonymous, then f will optimally employ a cutoff strategy. Furthermore, the strategic complements result of Proposition 4 continues to hold. Therefore, the existence of symmetric equilibria in pure strategies with the smallest and largest cutoffs follows from Theorem 5 of Milgrom and Roberts (1990) (see Appendix A for details).

Theorem 4 *There exists a symmetric equilibrium in pure cutoff strategies where 1) workers signal to their most preferred firms and 2) firms use symmetric cutoff strategies. There exist pure symmetric equilibria with the smallest and largest cutoffs.*

Note that the above theorem states the existence of a symmetric equilibrium in pure strategies, compared to Theorem 2 that established the existence of mixed strategy equilibria.

In a single-block case, we have additional results regarding the welfare of firms. Firm f responding more to signals has a negative effect on the welfare of other firms.

Proposition 7 (Negative Spillover) *Assume firms use cutoff strategies and workers signal their most preferred firms. Fix the strategies of firms $-f$ as s_{-f} . Let firm f 's strategy s_f differ from \tilde{s}_f only in that \tilde{s}_f has weakly greater cutoffs (responds more to signals). Then for all $f' \in -f$, we have $\mathbb{E}_\theta[\pi_{f'}(s_f, s_{-f}, \theta)] \leq \mathbb{E}_\theta[\pi_{f'}(\tilde{s}_f, s_{-f}, \theta)]$.*

The intuition of this result is that when firm f makes an offer to its top signaled worker, this worker accepts its offers with a probability equal one. However, when firm f makes an offer to its top-ranked worker, this creates less competition to other firms, since its top-ranked worker may prefer other firms to firm f . Note that this intuition does not hold in the multi-block case, because there is no guarantee that offers to workers who have signaled to the firm will be accepted.

Consider any two cutoff equilibria in which one equilibrium has greater cutoffs. As a corollary of Proposition 5, Proposition 6, and Proposition 7, both the expected number of matches and the welfare of workers will be weakly greater in the equilibrium with greater cutoffs, whereas the welfare of firms is weakly smaller.

Theorem 5 *Consider any two symmetric cutoff strategy equilibria where one equilibrium has greater cutoffs. Compared to the equilibrium with lower cutoffs, in the equilibrium with greater cutoffs,*

- *the expected number of matches is weakly greater,*
- *workers have weakly higher expected payoffs, and*
- *firms have weakly lower expected payoffs.*

The result states that in markets with a single block, firms and workers are *opposed* in their preferences over equilibria. When multiple equilibria exist, workers prefer the equilibrium that involves the greatest cutoffs (firms respond most to signals) while firms prefer the equilibrium with the lowest cutoffs. This result may seem unusual, in that it appears to indicate that firms prefer less signaling to more. However, note that this comparison only holds across equilibria in the presence of a signaling mechanism. Firms may or may not prefer any of these equilibria to the equilibrium of the offer game with no signals.

6 Market Structure And The Value of a Signaling Mechanism

In this section, we analyze how the expected increase in the number of matches from the introduction of a signaling mechanism, a measure we term the *value of the signaling mechanism*, differs across market structures. For the purpose of exposition clarity, we postpone the precise formulation of assumptions and most of the propositions to Appendix C.

To isolate the impact of a signaling mechanism on the number of matches in the market, we consider the *pure coordination model*, wherein firms and workers want to match, but are *almost* indifferent over the identity of the match. Specifically, we consider the cardinal utility from being matched to a partner as being *almost* the same across partners. If agent a has a preference profile θ_a , it prefers to be matched with partner θ_a^k , rather than with partner $\theta_a^{k'}$, $k' > k$, though the difference between utility intensities is very small (see Appendix C for details).

In addition, there is only one block of firms, i.e., worker preferences are uniformly distributed. Under these assumptions, there is the unique non-babbling symmetric equilibrium that satisfies criterion D1 of Cho and Kreps in the offer game with signals. Each worker sends a signal to her top-preferred firm. Each firm makes an offer to its top-signalized worker if it receives at least one signal; otherwise, it makes an offer to its top-ranked worker (see Proposition C1). As a consequence of Theorem 1, there is also the unique equilibrium of the offer game with no signals.

We define the expected number of matches in the unique equilibrium in the pure coordination model with signals (with no signals) and with F firms and W workers as $S(F, W)$ ($U(F, W)$). Define the expected increase in matches from the introduction of the signaling mechanism as $D(F, W) \equiv S(F, W) - U(F, W)$. We call $D(F, W)$ the *value of the signaling mechanism*.

6.1 Balanced Markets

In this subsection, we analyze how the value of the signaling mechanism changes for markets of various sizes. Fixing the equilibrium behavior in the offer games with and with no signals, described above, we calculate $D(F, W)$ for these markets. Figure 1 graphs $D(F, W)$ as a function of F for fixed $W = 10$ and $W = 100$, and $D(F, W)$ as a function of W for fixed $F = 10$ and $F = 100$.

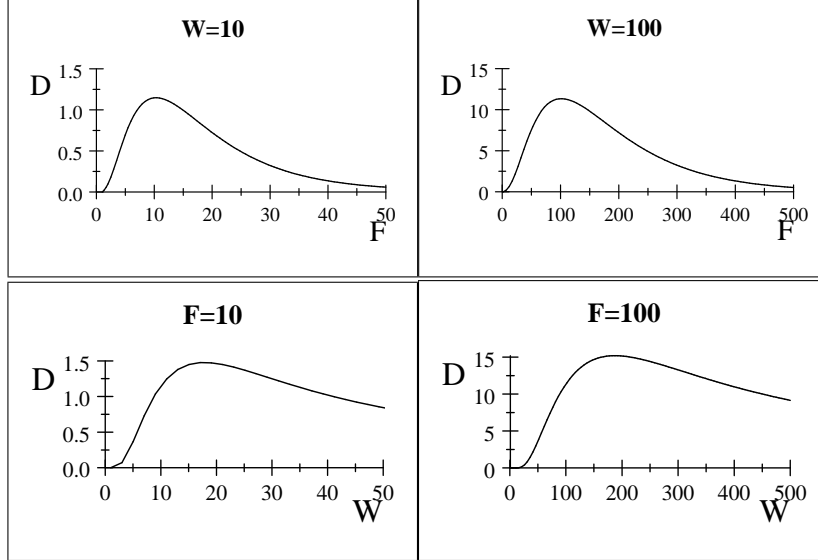


Figure 1. Balanced markets.

The single-peakedness of the graphs suggests that the value of the signaling mechanism is greatest for balanced markets—markets wherein the number of firms and the number of workers are of the same magnitude. To understand why signaling is most useful in balanced markets, it is helpful to think about the endpoints. With many workers and very few firms, firms will almost certainly match with or without the signaling mechanism. With many firms and few workers, the reverse holds: most workers will get offers with or without the signaling mechanism. Hence, the signaling mechanism offers little benefit at the extremes.

Furthermore, the graphs suggest that the percentage increase in the expected number of matches remains steady as market size increases, holding constant the ratio of workers to firms. Proposition 8 shows these observations precisely.

Proposition 8 (Balanced markets) *For fixed W , $D(F, W)$ attains its maximum value at $F = x_0W + O_W(1)$, where $x_0 \approx 1.01211$ is a constant and $O_W(1)$ is a function that is smaller than a constant for large W . For fixed F , $D(F, W)$ attains its maximum value at $W = y_0F + O_F(1)$, where $y_0 \approx 1.8842$ is a constant and $O_F(1)$ is a function that is smaller than a constant for large F .*

The proof of Proposition 8 proceeds by the calculation of an explicit formula for $D(F, W)$. The proof shows that the expected increase in the number of matches can be represented as

$$D(F, W) = F\alpha\left(\frac{W}{F}\right) + O_F(1)$$

or as

$$D(F, W) = W\beta\left(\frac{F}{W}\right) + O_W(1)$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are some functions. Hence, $D(F, W)$ is “almost” homogeneous of degree one for large markets. Therefore, we can evaluate the introduction of the signaling mechanism for a sample market, and its properties will be preserved for markets of other sizes, but with the same ratio of firms to workers.

For example, we can use Figure 1 to investigate maximal quantitative gains from the introduction of the signaling mechanism in large markets. For a fixed number of workers, the maximum increase is 11% of the total number of firms, whereas for a fixed number of firms the maximum increase is 15% of the total number of workers. Furthermore, the returns to the signaling mechanism are substantial over a wide range of market conditions. For example, only when the number of firms outweighs the number of workers by more than fivefold do the gains to the signaling mechanism drop to below 2%.

6.2 Markets with many periods of interactions

We now consider markets wherein agents can interact in multiple periods. Each worker can send only one signal and has only one interview position to fill. There are $L + 1$ periods in the offer game with signals. Workers send signals to firms in period 0. The other L periods, period 1 through period $L + 1$, agent can interact to be matched.

Period 0. Workers send signals.

1. Agent preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.

Periods 1 – L : Agents Interact. Each period consists of two stages:

1. Each firm makes an offer to at most one worker; offers are made simultaneously.
2. Each worker may accept at most one offer from the set of offers she receives.

Offers are binding, and workers can hold offers from period to period. When a worker accepts an offer, the firm-worker pair leaves the market, and this is observed by all agents. The other agents participate in the remaining periods. As a point of comparison, we will also be interested in the L -period offer game with no signals, which is identical to the game described above except that period 0 is excluded. We also presume that there is a discount factor $\delta \in (0, 1)$ for being matched in the latter periods.

Under the assumptions of the pure coordination model and a small discount factor, there is no incentive to delay offers or offer acceptance as agents care only about being matched. There is a unique symmetric sequential equilibrium in the offer game with no signals, where each firm makes an offer to its most preferred worker and each worker accepts its best offer

in each period. Similarly, there is the unique symmetric sequential non-babbling equilibrium that satisfies criterion D1 of Cho and Kreps in the offer game with signals. Each worker sends her signal to her top firm at period 0. Each worker accepts the best available offer at each period. Each firm makes an offer to its top signaled worker among the workers in the market; otherwise the firm makes an offer to its top-ranked worker among the workers in the market (see Proposition C3).

Now we are able to compare the offer game with and without the signaling mechanism for many different periods of interaction. Additional periods of interaction provide an opportunity for a greater number firms and workers to be matched. Therefore, the value of the signaling mechanism decreases as the number of periods of interaction increases. The next proposition shows this relationship formally. Since its proof is intuitive, we also provide it here.

Proposition 9 *The expected increase in the number of matches from the introduction of the signaling mechanism, $D_{1,1}^L(F, W)$, is a decreasing function of the number of periods of interactions, L .*

Proof.

For clarity of the argument, we compare markets with one and two periods of interaction. When there is a single period, signals increase the expected number of matches. In markets with two periods, all pairs matched in the first period leave the market. Therefore, the expected number of remaining market participants in the second period is *greater* in the offer game with no signals than it is in the offer game with signals.

Observe that when workers can send only one signal, all firms that receive at least one signal leave the market in period 1 (since in this case, signals indicate that offers will be accepted). Therefore, no firms, remaining in period 2, have received signals, so the second period of the offer game with signals is identical to a single period offer game with no signals. As Proposition 8 shows, the number of matches in a market with one period is proportional to the size of the market. Therefore, the expected number of matches in the second period in the market with no signals is greater than in the market with signals. Hence, the *difference* between the expected number of matches in the offer game with signals and the offer game with no signals decreases upon adding a second period of interaction. This logic extends to L periods of interaction. \square

Using simulations, we investigate how the value of the signaling mechanism declines with the introduction of additional periods of interaction for balanced markets. Figure 2 graphs the results for markets with $F = 100$ firms, $W = 100$ workers and $L = 1, \dots, 5$ periods of interaction.

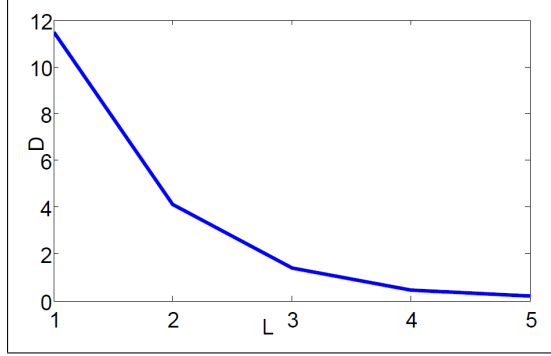


Figure 2. Several periods of interaction.

Note that for markets with the same number of firms and workers the increase in the expected number of matches due to the introduction of the signaling mechanism decreases by 55% when a second period is added, and decreases by 85% when second and third periods are added. Moreover, the value of the signaling mechanism is less than 0.5% of the total number of agents in markets with four periods of interaction.

6.3 The optimal number of signals and the number of interviews

In this subsection we analyze the markets in which workers can send several identical signals and have several positions, which we call *interviews*, to fill. Each worker can send up to K signals and has I interview positions. We assume $F > K \geq 1$ and $F > I \geq 1$, and each worker utility function is additive with respect to the number of interviews. In addition, we maintain the assumptions of the pure coordination model and that firms can only make up to one offer. The timing is the same as in our base model in Section 2.

Similar to previous analysis, there is the unique equilibrium in the offer game with no signals: each firm f makes an offer to TRW_f , and each worker accepts the best I offers among those she receives. There is also the unique symmetric non-babbling equilibrium that satisfies criterion D1 of Cho and Kreps in the offer game with signals: each worker signals to her most preferred K firms; each firm f makes an offer to TSW_f , if it receives at least one signal, and to TRW_f if it receives no signals. Again, each worker accepts the best I offers among those available (see Proposition C2).

Denote as $D_{I,K}(F, W)$ the expected increase in the number of matches from the introduction of the signaling mechanism when each worker has I interview positions and can send up to K signals.

Fixing the equilibrium behavior of agents, described above, we use simulations to analyze $D_{I,K}(F, W)$. Since the value of the signaling mechanism is the greatest for balanced markets, we analyze markets in which the number of firms equals the number of available positions. Figure 3 shows the results of simulations for markets with $W = 50$ workers, $I = 1$, $I = 2$,

and $I = 3$ interviews, and $F = 50$, $F = 100$, and $F = 150$ firms correspondingly (solid, dashed, and dot-dashed lines).

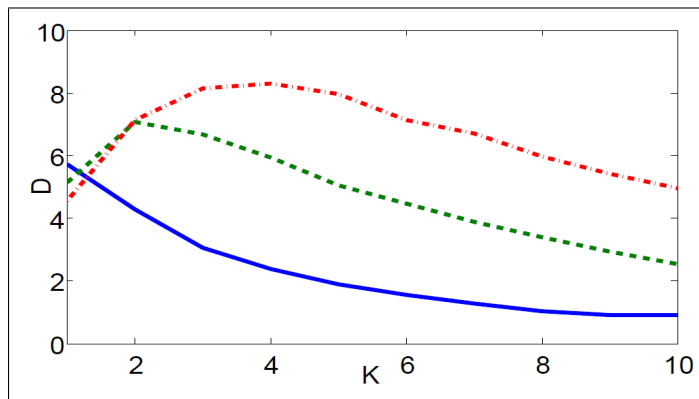


Figure 3. Multiple signals and interview positions.

The simulations demonstrate that if each worker has only one interview, then $D_{1,K}(F, W)$ is a decreasing function of K . However, the optimal number of signals may be greater than one if workers can have many interviews. The simulations also suggest that the value of the signaling mechanism is single-peaked in the number of signals as well as that the optimal number of signals is an increasing function of I .

7 Conclusion

Many real markets are overcrowded with applications. At the same time, employers need to incur costs in order to evaluate potential candidates. This leads to *congestion*. In many cases, this congestion is accompanied by informal *preference signaling*. In a smaller and perhaps newer set of markets, formal signaling mechanisms have emerged.

We have examined a natural signaling mechanism; allowing workers to send costless signals to a finite set of firms. While participation is voluntary, this mechanism nevertheless provides workers with a means of expressing preferences that is both credible and equitable. In a setting where workers agree on the ranking of blocks of firms but vary in their preferences within each block, workers will signal to their most preferred firm within each block. Firms use this information as guidance, optimally using cutoff strategies to make offers. We find that on average, introducing a signaling technology increases both the expected number of matches as well as the expected welfare of workers. The welfare of firms, on the other hand, changes ambiguously, because firms responding more to signals imposes a negative externality on other firms.

In addition, we find that the signaling mechanism adds the most value for markets in which the number of firms and the number of workers are of roughly the same magnitude. The optimal number of signals—the number of signals that maximizes the expected increase in the number of matches—may be greater than one for markets wherein each worker could have many positions to fill. We also show, by way of simulations, that the optimal number of signals increases when workers have more interviews. Finally, the expected increase in the number of matches from the introduction of the signaling mechanism decreases as the number of periods of interaction increases.

We emphasize that signaling mechanisms have the potential to improve outcomes in congested markets in a reasonably non-invasive manner. As opposed to a central clearinghouse (as in the National Resident Matching Program (Roth and Peranson, 1999)), a centralized signaling mechanism is significantly less drastic. As a consequence, market designers may find it easier to get consensus from participants to introduce such a mechanism, which nevertheless may provide substantial benefit.

Our aim was to examine the value that a signaling mechanism might offer congested markets, and how that value varies across market structures and signal designs. We hope that this paper will inform policy for a broad set of imperfectly functioning markets that may need a nudge, but are not in need of electro-shock therapy. Our hope is that the approach in this paper will serve as a tool and as a benchmark; a framework for examining settings with alternative market assumptions, and a point of comparison for alternative signaling mechanisms.

A Appendix. Block-correlated preferences.

Proof of Theorem 1.

Let us fix some anonymous strategies of firms $-f$ and find an optimal strategy of firm f . Since, firms use anonymous strategies and their preferences are uniformly distributed, symmetry ensures that the probability that a worker accepts firm f 's offer is the same across workers. Hence, firm f 's optimal strategy is to make an offer to its top-ranked worker.

Let us show this more formally. We consider some realization of agent preference profile $\theta \in \Theta$. We denote $m_f(s_f, s_{-f}, \theta)$ be the probability of firm f being matched when it uses strategy s_f and the other agents use strategies s_{-f} (possibly mixed) and the realized agents preference profile is $\theta \in \Theta$. We compare two strategies of firm f : making an offer to its top worker $s_f = \theta_f^1 = w$ and making an offer to its n th top worker $s'_f = \theta_f^n = w^n$, $n > 1$.

We denote a permutation that changes the ranks of w and w^n in a firm preference profile as

$$\sigma : (\dots, w, \dots, w^n, \dots) \longrightarrow (\dots, w^n, \dots, w, \dots)$$

Let us consider a profile of preferences $\theta' \in \Theta$ such that $\theta'_f = \theta_f$, the ranks of workers w and w^n are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f, \theta'_{f'} = \sigma(\theta_{f'}),$$

worker w and worker w^n preference profiles are exchanged

$$\theta'_w = \theta_{w^n}, \theta'_{w^n} = \theta_w,$$

and $\theta_{w'} = \theta'_{w'}$ for any other $w' \in W \setminus \{w, w^n\}$.

Since, firm $-f$ strategies are anonymous we have that

$$s_{-f}(\theta'_{-f}) = s_{-f}(\sigma(\theta_{-f})) = \sigma(s_{-f}(\theta_{-f}))$$

This means that the probability of firm f' , $f' \in -f$, making an offer to worker w for profile θ equals the probability of making an offer to worker w^n for profile θ' . Moreover, since we exchange worker w and w^n preference lists for profile θ' , whenever it is optimal for worker w to accept firm f offer for profile θ , it is optimal for worker w^n to accept firm f 's offer for profile θ' . Therefore,

$$m_f(s_f, s_{-f}, \theta) = m_f(s'_f, s_{-f}, \theta')$$

Therefore, for each θ_{-f} there exists θ'_{-f} such that the probability of getting an offer from the top worker equals the probability of getting an offer from n th top worker. Moreover θ'_{-f}

is different for different θ_{-f} by the construction. Since, θ_{-f} and θ'_{-f} are equally possible

$$E_{\theta_{-f}} m_f(s_f, s_{-f}, \theta | \theta_f) = E_{\theta'_{-f}} m_f(s'_f, s_{-f}, \theta | \theta_f)$$

and

$$E_{\theta} m_f(s_f, s_{-f}, \theta) = E_{\theta} m_f(s'_f, s_{-f}, \theta)^9$$

Therefore, the expected probability of getting a match from firm f 's top choice equals the expected probability of getting a match from firm f 's n th top choice. Since, the utility from obtaining a former match is greater, the strategy of firm f of making an offer to its top worker is optimal. \square

Proof of Proposition 1.

We first define criterion $D1$ of Cho and Kreps in our setting¹⁰. Let us consider some block-symmetric sequential equilibrium. We fix strategies of agents except worker w and firm f , which we denote as $s_{-f,w}$. We also fixed the beliefs of firms other than firm f , which we denote as μ_{-f} . Then, we analyze strategies of worker w and strategies and beliefs for firm f . Worker w sends some off-equilibrium message to firm f whenever she sends a signal when the equilibrium strategy prescribes zero probability of sending a signal to firm f or she does not send a signal when the equilibrium strategy prescribes sending a signal to firm f with probability equal 1. According to the definition of anonymous strategies, the latter one can happen in block-symmetric equilibrium only if firm f is the only one firm in its block. However, the symmetry of workers strategies ensures that all workers send their signals to firm f with probability 1 in this case. Since, signals do not transmit information about workers type, this equilibrium is outcome equivalent to babbling equilibrium. We further concentrate on the first type of off-equilibrium messages.

Consider worker w that sends some off-equilibrium signal to firm f . We denote the expected equilibrium payoff of firm f as u_f^* and the expected equilibrium payoff of worker w as u_w^* . For each firm f 's type $\bar{\theta} \in \Theta_f$ and each set of signals that firm f could receive, $\bar{h} \subset W$, we denote the mixed best response of firm f that has beliefs $\bar{\mu}$ as

$$MBR_f(\bar{\theta}, \bar{h} \cup w, \bar{\mu}) = \arg \max_{s_f \in \Delta^W} E_{\theta_{-w}}(\pi_f(s_f, s_{-f}, \theta) | \theta_f = \bar{\theta}, h_f = \bar{h} \cup w, \mu_f = \bar{\mu}).$$

Then, we denote the mixed best response of firm f for its types and any possible profiles of

⁹Note that we analyze only anonymity strategies of firms. Therefore, the change of firm f strategy for only one realization of preference is not legitimate. A legitimate change is a change for the realization of any strategy profile.

¹⁰See Cho and Kreps (1987) for the original definition.

signals it may receive, conditional on receiving worker w 's signal as

$$MBR_f(w, \bar{\mu}) = \{MBR_f(\bar{\theta}, \bar{h} \cup w, \bar{\mu}) \text{ for all } \bar{\theta} \in \Theta_f, \bar{h} \subset W\}$$

We denote the set of best responses by firm f to probability assessments concentrated on set $\Omega \subset \Theta_w$ as

$$MBR_f(w, \Omega) = \bigcup_{\{\mu_f: \mu_f(\Omega)=1\}} MBR_f(w, \mu_f)$$

We also denote for any worker's type $t \in \Theta_w$

$$\begin{aligned} D_t &= \{\phi \in MBR_f(w, \Theta_w), u_w^*(t) < E_{\theta_{-w}}(\pi_w(s_w, \phi, s_{-w, f}, \theta) \mid \theta_w = t)\} \\ D_t^0 &= \{\phi \in MBR_f(w, \Theta_w), u_w^*(t) = E_{\theta_{-w}}(\pi_w(s_w, \phi, s_{-w, f}, \theta) \mid \theta_w = t)\} \end{aligned}$$

Intuitively, set D_t (D_t^0) is the set of firm f strategies such that worker w of type t , receives an expected payoff that is greater than (equal to) the equilibrium one. We say that the type t may be pruned from firm f 's beliefs if firm f 's off-equilibrium beliefs should put zero probability on worker w 's type t upon receiving a signal from her. Using the above notations criterion $D1$ in our setting can be stated as follows.

Criterion D1. Fix strategies of workers $-w$ and strategies and beliefs of firms $-f$. If for some worker w 's type $t \in \Theta_w$ there exists a second worker w 's type $t' \in \Theta_w$ with $D_t \cup D_t^0 \subseteq D_{t'}$, then t may be pruned from the domain of firm f 's beliefs.

The intuition behind this criterion is that whenever type t of worker w either wishes to defect and send an off-equilibrium signal to firm f or indifferent, some other type t' of worker w strictly wishes to defect. When we prune t of worker w from firm f 's beliefs, we believe that firm f puts infinitely more likely probability that off-equilibrium signal has come from type t' than from type t .

We first show that there cannot be a block-symmetric sequential equilibrium that satisfies criterion $D1$ such that the ex-ante probability of receiving an offer by some worker from a firm within block \mathcal{F}^b , $b \in \{1, \dots, B\}$, is smaller when the worker sends her signal to this firm compare to the case when she does not send her signal to this firm, i.e. $q^b < p^b$.

Let us assume that such block-symmetric sequential equilibrium exists. If there are at least two workers, agents use anonymous block-symmetric strategies, and agents' types are uncorrelated, each worker is unmatched with positive probability. Then, it is optimal for each worker to avoid sending her signal to any firm within block \mathcal{F}^b on the equilibrium path (she want to increase the probability of being matched to each firm). Therefore, some firm $f \in \mathcal{F}^b$ can receive a signal from some worker w only on off-equilibrium path. If it is beneficial for some type $\theta_w \in \Theta_w$ to deviate from the equilibrium path and send her signal to firm f

(in this case firm f makes an offer to worker w), then it is beneficial for any type of worker w , $\theta'_w \in \Theta_w$, such that firm f is the top firm within block \mathcal{F}^b , to deviate. Therefore, the only types (preference profiles) of worker w that are not pruned in firms beliefs according to criterion $D1$ are such that firm f is the top firm within block \mathcal{F}^b for worker w . Therefore, if it is optimal for firm f to make an offer to worker w when it does not receive her signal, it is optimal for firm f to make an offer to worker w when it receives her signal. This means that $q^b < p^b$ cannot be part of block-symmetric sequential equilibrium that satisfies $D1$.

As a consequence of the above argument, we have that for any $b = 1, \dots, B$ $q^b \geq p^b$. It is easy to observe, that there is exist a block-symmetric sequential equilibrium that satisfies criterion $D1$, when for any $b = 1, \dots, B$, $q^b = p^b$. For example, each worker uses the strategy that prescribes sending her signal to firms with equal probability independently on their preferences and firms play the equilibrium strategies of the offer game with no signals. The equilibrium beliefs are block-uniform, i.e. if firm receives a signal from worker w its beliefs coincide with the priors. As one could see there are no off-equilibrium paths need to be specified. Therefore, this sequential equilibrium satisfies criterion $D1$.

Let us now consider the case when there exists $b_0 \in \{1, \dots, B\}$, such that $q^{b_0} > p^{b_0}$ in some block-symmetric sequential equilibrium. Recall that the probability that a worker sends her signal to firms within block \mathcal{F}^b is denoted as α_b , and we have that $\alpha_b \in [0, 1]$ and $\sum_{b=1}^B \alpha_b \leq 1$. Let us consider some block \mathcal{F}^b , such that $\alpha_b > 0$. As we mentioned above, if there are at least two workers, agents use anonymous block-symmetric strategies, and agents' types are uncorrelated, each worker is unmatched with positive probability in equilibrium. Therefore, $\alpha_b > 0$ and $q^b = p^b$ are incompatible in an equilibrium (worker w can benefit by signaling to block \mathcal{F}^{b_0} rather than block \mathcal{F}^b). Then, if $q^b > p^b$ worker w should send her signal to her top firm within block \mathcal{F}^b , as it delivers the greatest expected payoff to her.

Let us now consider some block $\mathcal{F}^{b'}$, $b' \in \{1, \dots, B\}$, such that $\alpha_{b'} = 0$, and some firm $f \in \mathcal{F}^{b'}$ that receives a signal on off-equilibrium path from some worker w . Therefore, either there exists type $t \in \Theta_w$ of worker w such that $D_t \neq \emptyset$ or for any type $t \in \Theta_w$ we have that $D_t = \emptyset$. For the former case, if worker w sends a signal to firm f , firm f offer delivers expected payoff to worker w of type t greater than the equilibrium one. However, whenever firm f offer delivers greater expected payoff than equilibrium one to worker w of type t , it delivers greater expected payoff than equilibrium one to worker w of type t' , which prefers firm f to any other firm in block $\mathcal{F}^{b'}$. Therefore, the only firm f off-equilibrium beliefs that survive criterion $D1$ are such that

$$\mu_f(\{\theta_w \in \Theta_w : f = \arg \max_{f' \in \mathcal{F}^{b'}} v(\theta_w, f')\} | w \in h_f) = 1 \quad (1)$$

Since, $D_{t'}$ and $D_{t'}^0$ consist of firm f 's best responses, it is optimal for firm f to make an offer to worker w upon receiving her signal if one restricts her beliefs to (1). This means that the

equilibrium strategy of worker w of type t' is not optimal if firm f has beliefs (1). Therefore, there cannot exist type $t \in \Theta_w$ of worker w such that $D_t \neq \emptyset$ in block-symmetric sequential equilibrium that satisfies criterion $D1$.

Let us now consider the case when for any type $t \in \Theta_w$ we have that $D_t = \emptyset$. Therefore, it is not beneficial for any type of worker to send an off-equilibrium signal. Therefore, $\alpha_{b'} = 0$ is an equilibrium strategy of worker w independently on off-equilibrium beliefs of firm f . Worker w strategy is optimal for any off-equilibrium beliefs of firms from block $\mathcal{F}^{b'}$, even if each firm f has most favorable for worker w beliefs, such as in (1).

Note that if there are at least two workers, the interaction between worker w and some firm f (fixing the strategies and beliefs of other agents) is a monotonic signaling game of Cho and Sobel (1990). The assumption of monotonicity is satisfied in our environment because each type of worker w prefers the same action of firm f , i.e. firm f making an offer to worker w . As a consequence, criterion $D1$ is equivalent to “never a weak best response” of Cho and Kreps (1987) and “universal divinity” of Banks and Sobel (1987) in our setting. More detailed discussion of monotonic signaling games can be found in Cho and Sobel (1990). \square

Proof of Proposition 2.

Let us consider firm f from some block \mathcal{F}_b , $b \in \{1, \dots, B\}$ that has the realized preference profile $\theta^* \in \Theta_f$ and that receives signals from the set of workers $h \subset W$. We denote worker TSW_f as w and some other worker from h as w' . We first prove that the expected payoff to f from making an offer to worker w is strictly greater than the expected payoff from making an offer to worker w' . We denote the corresponding strategies of firm f as $s_f(\theta^*, h) = w$ and $s'_f(\theta^*, h) = w'$.

According to Proposition 1, firm f believes that it is the top firm within block \mathcal{F}_b in workers w and w' preference lists. Let us denote the set of possible agents profiles consistent with firm f beliefs as

$$\bar{\Theta} \equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \arg \max_{f' \in \mathcal{F}^b} v(\theta_w, f) \text{ for each } w \in h\}$$

Similar to the proof of Theorem 1, we denote a permutation that changes the ranks of w and w' in a firm preference profile as

$$\sigma : (\dots, w, \dots w', \dots) \longrightarrow (\dots, w', \dots w, \dots)$$

Let us consider a profile of agents preferences $\theta' \in \Theta$ such that $\theta'_f = \theta^*$, the ranks of workers w and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f, \theta'_{f'} = \sigma(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'}, \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in W \setminus \{w, w'\}$, $\theta_{w^0} = \theta'_{w^0}$. Since, firm f preference list is unchanged, $\theta'_f = \theta^*$, and firm f has symmetric beliefs about workers w and w' types, profile θ' belongs to $\bar{\Theta}$. Since, firm $-f$ strategies are anonymous, for any $f' \in -f$ for any $h^0 \subset W$

$$s_{f'}(\sigma(\theta_{f'}), \sigma(h^0)) = \sigma(s_{f'}(\theta_{f'}, h^0))$$

Worker w and w' send their signals to firm f for both profiles θ and θ' . Therefore, they do not send their signals to firms $-f$, $\sigma(h^0) = h^0$. Since $\theta'_f = \sigma(\theta_f)$ we have that

$$s_{f'}(\theta'_{f'}, h^0) = \sigma(s_{f'}(\theta_{f'}, h^0))$$

This means that the probability of firm f' making an offer to worker w for profile θ equals the probability of making an offer to worker w' for profile θ' . Moreover, since we exchange worker w and w' preference lists for profile θ' , whenever it is optimal for worker w to accept firm f offer for profile θ , it is optimal for worker w' to accept firm f' 's offer for profile θ' . Since firms types are independent, the probability of firm f being matched when it uses strategy s_f for profile θ equals the probability of firm f being matched when it uses strategy s'_f for profile θ'

$$m_f(s_f, s_{-f}, \theta) = m_f(s'_f, s_{-f}, \theta').$$

Therefore, for each $\theta \in \bar{\Theta}$ there exists $\theta' \in \bar{\Theta}$ such that the probability that firm f gets an offer from worker w equals the probability that firm f gets an offer from worker w' . Moreover, profile θ' is different for different θ by the construction. Since, θ and θ' are equally possible

$$E_\theta m_f(s_f, s_{-f}, \theta | \theta \in \bar{\Theta}) = E_\theta m_f(s'_f, s_{-f}, \theta | \theta \in \bar{\Theta})$$

Therefore, the expected probability that firm f gets a match if it makes an offer to some worker in h is the same across all workers in h . Since, the utility from obtaining a match from TSW_f is the largest, the expected payoff to f from making an offer to TSW_f is strictly greater than the payoff from making an offer to any other worker in h .

A similar construction is valid for the workers in set $W \setminus h$. Therefore, the probability that firm f 's offer is accepted is the same across any worker in $W \setminus h$. Hence, firm f prefers making an offer to its most valuable worker - TRW_f - than to any other worker in $W \setminus h$ ¹¹.

¹¹Note that it can happen that $TRW_f = TSW_f$. In this case the statement of the proposition is still valid since firm f believes that it is the TRW_f 's top firm within block \mathcal{F}_b . Therefore, firm f prefers making an offer to TRW_f rather to any other worker in set W .

□

Proof of Proposition 3.

Let us first note that Proposition 1 and Proposition 2 established that the optimal choice of firm f for each set of received signals is either TSW_f , TRW_f , or some lottery between them, when the payoff from making an offer to TSW_f and TRW_f is the same. We break the proof into two parts. First we show that the identities of workers that have sent a signal to firm f influence neither the expected payoff of TSW_f nor TRW_f provided that the total number of signals firm f receives is constant and TSW_f does not change. Second we prove that if it is optimal for firm f to choose TSW_f when it receives signals from some set of workers, then it still optimal for firm f to choose TSW_f if the number of received signals does not change and TSW_f has a smaller rank (TSW_f is more valuable).

Let us consider some firm f from block \mathcal{F}_b , $b \in \{1, \dots, B\}$ and some realization of its preferences, $\theta_f = \theta^*$. Assume that it is optimal for firm f to make an offer to TSW_f if it receives set of signals $h \subset W$. We want to show that if firm f receives the set of signals h' , which has the same TSW_f and the same power, $|h'| = |h|$, it is still optimal for firm f to make an offer to TSW_f . We consider the case when h and h' differ only in one signal. There exist worker w and worker w' , such that the former one belongs to set h , but it does not belong to set h' ; while the latter one belongs to h' , but it does not belong to h . General case directly follows. We consider two firm f strategies

$$\begin{aligned} s_f(\theta^*, \cdot) &= TSW_f(\theta^*, \cdot) \\ s'_f(\theta^*, \cdot) &= TRW_f(\theta^*, \cdot) \end{aligned}$$

According to Proposition 1, firm f believes that it is the top firm within block \mathcal{F}_b for workers that send signals to it. We denote the set of possible agents' profiles consistent with these beliefs when firm f receives signals from h and h' as

$$\begin{aligned} \bar{\Theta}^h &\equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \arg \max_{f' \in \mathcal{F}^b} v(\theta_w, f') \text{ for each } w \in h\} \\ \bar{\Theta}^{h'} &\equiv \{\theta \in \Theta \mid \theta_f = \theta^* \text{ and } f = \arg \max_{f' \in \mathcal{F}^b} v(\theta_w, f') \text{ for each } w \in h'\} \end{aligned}$$

correspondingly.

We now construct a bijection between $\bar{\Theta}^h$ and $\bar{\Theta}^{h'}$. We denote a permutation that changes the ranks of w and w' in a firm preference profile as

$$\sigma : (\dots, w, \dots w', \dots) \longrightarrow (\dots, w', \dots w, \dots)$$

For any profile $\theta \in \bar{\Theta}^h$ we construct profile $\theta' \in \bar{\Theta}^{h'}$ such that $\theta'_f = \theta^*$, the ranks of workers w

and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f, \theta'_{f'} = \sigma(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'}, \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in W \setminus \{w, w'\}$, $\theta_{w^0} = \theta'_{w^0}$.

Since, firm f preference list is unchanged, $\theta'_f = \theta^*$, and firm f has symmetric beliefs about workers w for profile θ and about worker w' for profile θ' , profile θ' belongs to $\bar{\Theta}^{h'12}$. By construction, profile θ' is different for different θ . Since, the powers of sets $\bar{\Theta}^h$ and $\bar{\Theta}^{h'}$ are the same, the above correspondence is a bijection.

Since, firm $-f$ strategies are anonymous, for any $f' \in -f$ and $h^0 \subset W$

$$s_{f'}(\sigma(\theta_{f'}), \sigma(h^0)) = \sigma(s_{f'}(\theta_{f'}, h^0))$$

This means that the probability of firm f' making an offer to worker w for profile θ equals the probability of firm f' making an offer to worker w' for profile θ' . Moreover, since we exchange worker w and w' preference lists for profile θ' , whenever it is optimal for worker w to accept firm f offer for profile θ , it is optimal for worker w' to accept firm f 's offer for profile θ' . Since firms types are independent, the probability of firm f being matched when it uses strategy $s_f(\theta^*, \cdot)$ for profile θ equals the probability of firm f being matched when it uses strategy $s_f(\theta^*, \cdot)$ for profile θ'

$$m_f(s_f, s_{-f}, \theta) = m_f(s_f, s_{-f}, \theta')$$

Similar for strategy $s'_f(\theta^*, \cdot)$

$$m_f(s'_f, s_{-f}, \theta) = m_f(s'_f, s_{-f}, \theta')$$

Since, we constructed a bijection between $\bar{\Theta}^h$ and $\bar{\Theta}^{h'}$ and θ and θ' are equally likely

$$\begin{aligned} E_\theta m_f(s_f, s_{-f}, \theta | \theta \in \bar{\Theta}^h) &= E_{\theta'} m_f(s_f, s_{-f}, \theta' | \theta' \in \bar{\Theta}^{h'}) \\ E_\theta m_f(s'_f, s_{-f}, \theta | \theta \in \bar{\Theta}^h) &= E_{\theta'} m_f(s'_f, s_{-f}, \theta' | \theta' \in \bar{\Theta}^{h'}) \end{aligned}$$

Therefore, if firm f optimally makes an offer to TSW_f (TRW_f) for the set of signal h ,

¹²Note that we do not consider signaling strategy of workers. Since, workers can send signals to different firms for the same preference profile. We just want to establish one-to-one correspondence between $\bar{\Theta}^h$ and $\bar{\Theta}^{h'}$.

it also should optimally make an offer to TSW_f (TRW_f), which is the same worker, for the set of signals h' .

Now we prove that if firm f optimally chooses TSW_f for set of signals h , then it should still optimally choose TSW_f for set of signals h' , if the number of received signals is the same, $|h'| = |h|$, and TSW_f of set h' has a smaller rank (TSW_f is more valuable). We consider set h' that differs from h only in the best (for firm f) worker and the difference between the ranks of top signaled workers equals one. General case directly follows.

$$\begin{aligned} \text{for any } w \in h/TSW_f(\theta^*, h) &\Leftrightarrow w \in h'/TSW_f(\theta^*, h') \\ \text{rank}_f(TSW_f(\theta^*, h')) &= \text{rank}_f(TSW_f(\theta^*, h)) - 1 \end{aligned}$$

Note that sets \bar{h} and \bar{h}' differs only in one signal. The construction of the first part of the proof also works in this case. We consider sets of profiles and correspondence similar to the one above. Similar, we can show that the probabilities of firm f being matched with TSW_f (TRW_f) are the same for h and h' . However, now if firm f offer to TRW_f is accepted, firm f gets the same payoff ($u(1)$) for two different profiles. Whereas if firm f offer to TSW_f is accepted, firm f gets strictly greater payoff for set h' compare to set h , because TSW_f has smaller rank for the former case. Hence, if it is optimal for firm f to make an offer to TSW_f when it receives set of signals h , it is optimal for firm f to make an offer to TSW_f when firm f receives set of signals h' .

The two statements we have just proved allows us to conclude that if firm f use anonymous strategies, firm f 's optimal strategy could be represented as some cutoff strategy. Note that there can be other optimal strategies. If firm f is indifferent between making an offer to TSW_f and making an offer to TRW_f for some h , firm f could change its behavior (make offer to TSW_f or TRW_f) when it receives the sets of signals h' , which has the same TSW_f and the same power, $|h'| = |h|$. However, the strategy of keeping the same behavior for sets h and h' gives the same payoff to firm f . \square

Proof of Theorem 2.

We consider the set of cutoff strategies of firms and top-block strategies of workers that we denote as S . We denote its typical element as s , i.e. $s = (s_F, s_W) \in S$, that consists of firms cutoff strategies, $s_F = (s_{f_1}, \dots, s_{f_F})$, and workers top-block strategies strategies, $s_W = (s_{w_1}, \dots, s_{w_W})$.

A strategy of firm f , s_f , is a vector of real numbers of size W that specifies cutoff points for each positive number of signals firm f could receive, $s_f = (j_f^1, \dots, j_f^W)$, where j_f^l is a real number from interval $[1, W]$ for each $l = 1, \dots, W$. We denote the set of possible firm cutoff strategies as $T_f = [1, W]^W$.

A top-block strategy of worker w , s_w , is a vector of size B that specifies the probability

that she sends her signal to her top firm of specific block $s_w = (\alpha_w^1, \dots, \alpha_w^B)$, where $\alpha_w^b \geq 0$ for each $b = 1, \dots, B$ and $\sum_{b=1}^B \alpha_w^b \leq 1$. We denote the set of possible worker top-block strategies as $T_w = \{(\alpha^1, \dots, \alpha^B) : \alpha^b \geq 0 \text{ and } \sum_{b=1}^B \alpha^b \leq 1\}$.

Let us also denote the expected payoff of worker w when she uses top-block strategy s_w and the other agents use strategy s_{-w} as¹³

$$U_w(s_w, s_{-w}) = E_\theta(\pi_w(s_w, s_{-w}, \theta))$$

and the expected payoff of firm f when it uses strategy s_f and the other agents use strategy s_{-f} as

$$U_f(s_f, s_{-f}) = E_\theta(\pi_f(s_f, s_{-f}, \theta)).$$

We introduce function $g : S \rightarrow 2^S$ such that

$$g_a(s) = \arg \max_{\beta \in T_a} U_a(\beta, s_{-a})$$

for each $a \in W \cup F$.

The immediate consequence of the above definitions is that T_f and T_w are *non-empty*, *convex*, and *compact*. Also, $U_w(s_w, s_{-w})$ is a linear function of its first argument. Namely, if we denote the expected payoff of worker w from sending a signal to some block $b \in \{1, \dots, B\}$, given the strategies of agents s_{-w} , as $\Pi_b(-s_w)$ and worker w employs strategy $s_w = (\alpha_w^1, \dots, \alpha_w^B)$, worker w payoff equals

$$U_w(s_w, s_{-w}) = \sum_{b=1}^B \alpha_b \Pi_b(-s_w)$$

Therefore, $g_w(s)$ is a continuous correspondence with closed graph.

Let us now consider function $U_f(s_f, s_{-f})$. Similar, let us consider some realization of preference profile θ when firm f receives $|h|$ signals. Given the strategies of other agents s_{-f} , we denote the expected payoff of firm f from making an offer to TRW_f as Π_{TRW} , and the expected payoff of firm f from making an offer to TSW_f as Π_{TSW} . Then, the payoff of firm f from using cutoff strategy $j_{|h|}$, $s_f = (\dots, j_{|h|}, \dots)$, equals

$$\pi_f(s_f, s_{-f}, \theta) = \begin{cases} \Pi_{TRW} & \text{if } j_{|h|} \leq \text{rank}(TSW) - 1 \\ ([j_{|h|}] - j_{|h|})\Pi_{TSW} + (j_{|h|} - [j_{|h|}])\Pi_{TRW} & \text{if } \text{rank}(TSW) > j_{|h|} > \text{rank}(TSW) - 1 \\ \Pi_{TSW} & \text{if } j_{|h|} \geq \text{rank}(TSW) \end{cases}$$

where $[j_{|h|}]$ and $\lfloor j_{|h|} \rfloor$ denote the closest integer larger and smaller than $j_{|h|}$ correspondingly.

Function $\pi_f(s_f, s_{-f}, \theta)$ is quasi-concave function of cutoff $j_{|h|}$. Therefore, the expected

¹³Note that the strategy of agents are anonymous. Therefore, they do not depend on particular realization of preferences.

payoff from using cutoff $j_{|h|}$, $E_\theta[\pi_f(s_f, s_{-f}, \theta)|h_f = |h|]$, is also a quasi-concave function of cutoff $j_{|h|}$ as it is a linear combination of quasi-concave functions. Therefore, $U_f(s_f, s_{-f})$ is a quasi-concave function of its first argument. Therefore, $g_f(s)$ is a continuous correspondence with closed graph. Overall, we get that $g(s)$ is a continuous correspondence with closed graph. Hence, $g(s)$ has a fixed point by Kakutani theorem (see Kakutani, 1941).

We have considered above only the set of cutoff strategies. However, Proposition 1 and Proposition 3 allow us to conclude that the above equilibrium is an equilibrium when we allow any deviations, not only in cutoff strategies. Overall, we have established the existence of equilibrium when workers use symmetric top-block strategies and firms use symmetric cutoff strategies and have top-block beliefs. \square

Proof of Proposition 4.

We consider firm f from some block \mathcal{F}_b , $b \in \{1, \dots, B\}$. We consider two sets of other firms strategies, s_{-f} and s'_{-f} , that vary only in firm f' strategies. For simplicity, we assume that s_f differs from $s_{f'}$ only for profile $\bar{\theta}_{f'}$ and set of received signals $\bar{h}_{f'}$

$$\begin{aligned} s_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) &= \alpha TSW_{f'} + (1 - \alpha)TRW_{f'} \\ s'_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) &= \alpha' TSW_{f'} + (1 - \alpha')TRW_{f'} \end{aligned}$$

such that $\alpha' > \alpha$. Formally, this means either $s'_{f'}$ or $s_{f'}$ is not a cutoff strategy, because cutoff strategy should be the same for any profile of preferences (anonymity) and when firms receives the same number of signals. The extension to the change in firm f' cutoff strategy, i.e. for any profile of preferences and any set of signals of the same size, is immediate.

Let us consider some realization of firm f preference profile $\theta_f = \theta^*$ and some set of signals $h \subset W$. We denote firm f strategies of making an offer to TRW_f and TSW_f as

$$\begin{aligned} s_f(\theta^*, h) &= TRW_f \\ s'_f(\theta^*, h) &= TSW_f \end{aligned}$$

correspondingly. We want to show that firm f 's payoff from making an offer to TRW_f decreases whereas firm f 's payoff from making an offer to TSW_f increases when firm f' responds more to signals, i.e. plays strategy $s_{f'}$ instead of $s'_{f'}$.

$$\begin{aligned} I) E_\theta(\pi_f(s_f, s_{-f}, \theta)|\theta_f = \theta^*, h_f = h) &\geq E_\theta(\pi_f(s_f, s'_{-f}, \theta)|\theta_f = \theta^*, h_f = h) \\ II) E_\theta(\pi_f(s'_f, s_{-f}, \theta)|\theta_f = \theta^*, h_f = h) &\leq E_\theta(\pi_f(s'_f, s'_{-f}, \theta)|\theta_f = \theta^*, h_f = h) \end{aligned}$$

Since, firm f 's offer can only be either accepted or declined, the above statements are equiv-

alent to

$$\begin{aligned} I) E_\theta(m_f(s_f, s_{-f}, \theta)|\theta_f = \theta^*, h_f = h) &\geq E_\theta(m_f(s_f, s'_{-f}, \theta)|\theta_f = \theta^*, h_f = h) \\ II) E_\theta(m_f(s'_f, s_{-f}, \theta)|\theta_f = \theta^*, h_f = h) &\leq E_\theta(m_f(s'_f, s'_{-f}, \theta)|\theta_f = \theta^*, h_f = h) \end{aligned}$$

Namely, the probability of being matched to TRW_f decreases, and the probability of being matched to TSW_f increases.

I. We first prove the first statement. We define the sets of agents profiles that lead to the increase and decrease in the probability of getting a match given the change in firm f' strategy as

$$\begin{aligned} \bar{\Theta}_+ &\equiv \{\theta \in \Theta | \theta_f = \theta^*, h_f = h \text{ and } m_f(s_f, s_{-f}, \theta) < m_f(s_f, s'_{-f}, \theta)\} \\ \bar{\Theta}_- &\equiv \{\theta \in \Theta | \theta_f = \theta^*, h_f = h \text{ and } m_f(s_f, s_{-f}, \theta) > m_f(s_f, s'_{-f}, \theta)\} \end{aligned}$$

correspondingly. If set $\bar{\Theta}_+$ is empty, the statement has been proved. Otherwise, we take some $\theta \in \bar{\Theta}_+$ and denote $s_f(\theta^*, h) = TRW_f = w$. Then, we should have

$$\begin{cases} TRW_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) = w \\ TSW_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) = w' \neq w \end{cases} .$$

and

$$\begin{aligned} s_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) &= \alpha w' + (1 - \alpha)w \\ s'_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) &= \alpha' w' + (1 - \alpha')w \end{aligned}$$

Note that the situation when firm f is from a better group than firm f' cannot happen, i.e. $f' \in \mathcal{F}_{b'}$ where $b' > b$. In this case the offer of firm f' is always worse than the offer of firm f and cannot influence the probability that firm f obtains a match. Therefore, firm f is from a group that is weakly worse than $\mathcal{F}_{b'}$, i.e. $b' \leq b$.

Note that worker w has sent a signal neither to firm f nor to firm f' . This allows us to construct $\theta' \in \bar{\Theta}_-$. Similar to the proofs of the previous propositions we consider a permutation that changes the ranks of w and w' in a firm preference profile

$$\sigma : (\dots, w, \dots w', \dots) \longrightarrow (\dots, w', \dots w, \dots)$$

For any profile $\theta \in \bar{\Theta}_+$ we construct profile $\theta' \in \Theta$ such that $\theta'_f = \theta^*$, the ranks of workers w and w' are exchanged in the preference lists of firms $-f$

$$\text{for any firm } f' \in -f, \theta'_f = \sigma(\theta_f),$$

worker w and worker w' preference profiles are exchanged

$$\theta'_w = \theta_{w'}, \theta'_{w'} = \theta_w,$$

and for any other $w^0 \in W \setminus \{w, w'\}$, $\theta_{w^0} = \theta'_{w^0}$.

Preference profile θ' is such that $\theta_f = \theta^*$ and $h_f = h$. In addition, anonymity of firms strategies implies

$$\begin{aligned} s_{f'}(\theta'_{f'}, h'_{f'}) &= s_{f'}(\sigma(\theta_{f'}), \sigma(h_{f'})) \\ &= \alpha\sigma(w') + (1 - \alpha)\sigma(w) \\ &= \alpha w + (1 - \alpha)w' \end{aligned}$$

and

$$s'_{f'}(\theta'_{f'}, h'_{f'}) = \alpha'w + (1 - \alpha')w'$$

We assumed that firm f' prevents firm f from being matched with worker w for profile θ . Since, firm f' makes an offer to worker w , when it uses strategy $s'_{f'}$ rather than s_f , more frequently for profile θ' , the probability that firm f offer is accepted decreases. If firm f is from the group \mathcal{F}_b , $b > b'$, worker w always prefers firm f' to firm f . If firm f and firm f' are from the same group, $b = b'$, worker w has sent a signal to firm f' for profile of preferences θ' . This means that worker w prefers firm f' to firm f for profile θ' .

We should also investigate the behavior of a firm that receives worker w 's signal for profile θ , say firm f_y . If firm f_y makes an offer to worker w for profile θ , since the change of firm f' strategy changes firm f payoff, firm f_y should be worse than both firms f and f' in worker w preferences. Hence, firm f_y 's offer cannot change the incentives of worker w . If worker w' sends her signal to firm f_y then firm f_y either makes an offer to worker w' or to worker TRW_y not to worker w . Hence, firm f_y does not also change the incentives of the other agents. Overall, we conclude that $\theta' \in \bar{\Theta}_-$.

Note that the above construction gives for different profiles of $\bar{\Theta}_+$ different profiles of $\bar{\Theta}_-$. Therefore, we have constructed an injective function $\bar{\Theta}_+$ to $\bar{\Theta}_-$ and $|\bar{\Theta}_-| \geq |\bar{\Theta}_+|$ ¹⁴. Since, profiles θ and θ' are equally likely

$$E_\theta(m_f(s_f, s_{-f}, \theta)|\theta_f = \theta^*, h_f = h) \geq E_\theta(m_f(s_f, s'_{-f}, \theta)|\theta_f = \theta^*, h_f = h)$$

II. Let us now show that if firm f' responds more to signals, the probability of firm f being matched to TSW_f increases. If firm f , $f \in F_b$, receives a signal from worker w it believes that it is the best firm in group \mathcal{F}_b according to worker w preferences. Therefore,

¹⁴One may show that generally it is impossible to have a correspondence in the other direction.

worker w prefers the offer of firm f to an offer from any other firm f' from group $\mathcal{F}_{b'}$, $b' \geq b$. Therefore, the change of the behavior of any firm f' from group $\mathcal{F}_{b'}$, $b' \geq b$, does not influence firm f 's payoff.

If we consider any firm f' from group $\mathcal{F}_{b'}$, $b' < b$, it can draw away worker w 's offer from firm f only if it makes an offer to worker w . However, firm f' makes an offer to worker w , conditionally on firm f receiving worker w signals, only when worker w is $TRW_{f'}$. However, if firm f' responds more to signals, it makes an offer to its $TRW_{f'}$ more rarely. This means it is less often draw worker w away from firm f . Therefore, the probability that firm f offer is accepted by TSW_f increases.

$$E_\theta(m_f(s_f, s_{-f}, \theta) | \theta_f = \theta^*, h_f = h) \leq E_\theta(m_f(s_f, s'_{-f}, \theta) | \theta_f = \theta^*, h_f = h)$$

As a corollary of I) and II), if firm f' increases its cutoff point for some set of signals, firm f will also optimally (weakly) increase its cutoff points. Since, the above logic is valid for the change of cutoff points for any set of signals of the same size and any profile of preferences, the statement of the proposition immediately follows. \square

Proof of Proposition 5.

Let us consider firm f cutoff strategies s_f and s'_f such that s'_f has weakly greater cutoffs. Let us consider two sets

$$\begin{aligned} \bar{\Theta}_+ &\equiv \{\theta \in \Theta \mid m(s_f, s_{-f}, \theta) < m(s'_f, s_{-f}, \theta)\} \\ \bar{\Theta}_- &\equiv \{\theta \in \Theta \mid m(s_f, s_{-f}, \theta) > m(s'_f, s_{-f}, \theta)\} \end{aligned}$$

If profile θ is from set $\bar{\Theta}_+$, it must be the case that without firm f offer, TRW_f has an offer from another firm, and worker TSW_f does not

$$m(s'_f, s_{-f}, \theta) - m(s_f, s_{-f}, \theta) = 1. \quad (2)$$

Similarly, if profile θ is from set $\bar{\Theta}_-$, it must be the case that without firm f offer, TSW_f has an offer from another firm, and TRW_f does not

$$m(s'_f, s_{-f}, \theta) - m(s_f, s_{-f}, \theta) = -1. \quad (3)$$

We will now show that $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$. Equations (2) and (3) along with the fact that each $\theta \in \bar{\Theta}_+ \cup \bar{\Theta}_-$ happens equally likely will then be enough to prove the result.

Let us denote $TRW_f = w'$ and $TSW_f = w$. We construct function $\psi : \Theta \rightarrow \Theta$ as follows. Let $\psi(\theta)$ be the profile in which workers have preferences as in θ , but firms $-f$ all swap the

positions of workers w' and w in their preference lists. If profile θ belongs to $\bar{\Theta}_-$, without firm f 's offer, worker w has an offer from another firm, and worker w' does not. Therefore, when preferences are $\psi(\theta)$, without firm f 's offer two following statements should be true i) worker w' **must** have another offer and ii) worker w **cannot** have another offer.

To see i), note that under θ , worker w sends a signal to firm f , so his outside offer must come from some firm f' who has ranked him first. Under profile $\psi(\theta)$, firm f' ranks worker w' first. If worker w' has not sent a signal to firm f' , then by anonymity, w' gets the offer of firm f' . If worker w' has signaled to firm f' , worker w' again gets firm f' offer.

To see ii), suppose to the contrary that under $\psi(\theta)$, worker w does in fact receive an offer from some firm $f' \neq f$. Since worker w sends a signal to firm f , worker w must be $TRW_{f'}$ under $\psi(\theta)$, so that worker w' is $TRW_{f'}$ under θ . But then by anonymity w' receives the offer of firm f' under θ , a contradiction.

From i) and ii), we have

$$\theta \in \bar{\Theta}_- \Rightarrow \psi(\theta) \in \bar{\Theta}_+.$$

Since function ψ is injective, we have $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$. \square

Proof of Proposition 6.

Let us consider firm f cutoff strategies s_f and s'_f such that s'_f has weakly greater cutoffs. According to Proposition 5, the expected total number of matches increases when firm f responds more to signals (use strategy s'_f instead of s_f). Since, workers play symmetric strategies, the expected number of matches of worker w also increases. Using the construction of Proposition 5, one could show whenever worker w loses a match with firm f for profile θ when it responds more to signals (worker w is TRW_f) it is possible to construct profile θ' when worker w obtains the match (worker w is TSW_f). The function that matches these profiles is again injective. Moreover, worker w values more match when she is TSW_f rather when she is TRW_f . Therefore, ex-ante utility of worker w increases when firm f responds more to signals. \square

Proof of Theorem 3.

Let us denote firm strategies in the unique equilibrium of the offer game with no signals as s_F^0 . Now, consider a block-symmetric equilibrium of the offer game with signals when agents use strategies (s_F, s_W) . If agents employ strategies (s_F^0, s_W) , the expected number of matches and the welfare of workers equal the corresponding parameters in the offer game with no signals. Therefore, the result that the expected number of matches and the expected welfare of workers in a block-symmetric equilibrium in the offer game with signals are *weakly* greater than the corresponding parameters in the unique equilibrium of the offer game with no signals is a consequence of sequential application of Propositions 5 and 6.

Let us now consider some non-babbling block-symmetric equilibrium (s_F, s_W) of the offer game with signals such that there exists block \mathcal{F}_b with at least two firms such that $\alpha_b > 0$. Proposition 1 shows that firms from block \mathcal{F}_b should respond to signals in the equilibrium, $q^b > p^b$, i.e. make offers to top signaled workers with positive probability.

Let us take some firm f from block \mathcal{F}_b . Using a construction similar to the one in the proof of Proposition 5 we consider two sets

$$\begin{aligned}\bar{\Theta}_+ &\equiv \{\theta \in \Theta \mid m(s_f^0, s_{-f}, \theta) < m(s_f, s_{-f}, \theta)\} \\ \bar{\Theta}_- &\equiv \{\theta \in \Theta \mid m(s_f^0, s_{-f}, \theta) > m(s_f, s_{-f}, \theta)\}\end{aligned}$$

Let us consider some realized profile of preferences, $\theta \in \Theta$, and denote $TRW_f = w'$ and $TSW_f = w$. We also consider $\psi : \Theta \rightarrow \Theta$, such that $\psi(\theta)$ is the profile in which workers have preferences as in θ , but firms $-f$ all swap the positions of workers w' and w in their preference lists. Note that $\psi(\psi(\theta)) = \theta$ and ψ is a bijection on Θ . Proposition 5 shows that $|\bar{\Theta}_+| \geq |\bar{\Theta}_-|$. Let us now show there exist $\theta \in \bar{\Theta}_+$ such that $\psi(\theta) \notin \bar{\Theta}_-$.

There are at least two firms in block \mathcal{F}_b that responds to signals, which we denote as f and f' . Let us consider some profile θ from $\bar{\Theta}_+$. We again denote $TRW_f = w'$ and $TSW_f = w$. Therefore, worker w does not have an offer from any other firm for profile θ from $\bar{\Theta}_+$, but worker w' has at least two offers. Since worker w' sends her signal to firm f' with positive probability and firm f' responds to signals, i.e. makes offers to its top signaled workers, there exist $\theta^* \in \bar{\Theta}_+$ such that worker w' is top signaled worker of firm f' , and firm f' makes an offer to worker w' .

However, worker w for profile $\psi(\theta^*)$ does not have any other offer, because she is neither TRW_f nor TSW_f for profile $\psi(\theta^*)$. Therefore, $\psi(\theta^*)$ cannot belong to $\bar{\Theta}_-$. Therefore, we have found a profile from $\bar{\Theta}_+$ such that it does not belong to $\bar{\Theta}_-$. As a result, $|\bar{\Theta}_+| > |\bar{\Theta}_-|$ and we have that

$$E_\theta[m(s_f^0, s_{-f}, \theta)] < E_\theta[m(s_f, s_{-f}, \theta)]$$

In addition, we know that

$$E_\theta[m(s_f^0, s_{-f}^0, \theta)] \leq E_\theta[m(s_f^0, s_{-f}, \theta)],$$

which gives us

$$E_\theta[m(s_f^0, s_{-f}^0, \theta)] < E_\theta[m(s_f, s_{-f}, \theta)].$$

Overall, the expected number of matches in the offer game with signals when agents use strategies (s_F, s_W) is strictly greater than the expected number of matches in the offer game with no signals.

Using to the above construction and the logic of the proof of Proposition 6 one could get

the result for worker welfare.

Now, we show that signals can ambiguously influence the welfare of firms. We accomplish this goal by way of a simple example. Though the example presented below considers only the case of the uniform distribution of preferences (one block of firms) its result could be easily generalized for the block-uniform case.

Example A1 *Let us consider the market with two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. We compare the welfare of firms in the equilibria of two games: the offer game with no signals and the offer game with signals.*

i) Each firm makes its offer to its top worker in the unique equilibrium of the offer game with no signals. Hence, the expected welfare of each firm equals $U_f = \frac{3}{4}u(1)$.

ii) Let us consider a symmetric equilibrium of the offer game with signals. It can be characterized by a single cutoff point, when a firm receives one signal. If the firm receives zero or two signals it is optimal to make an offer to its top ranked worker (in a symmetric equilibrium). Similarly, if the firm receives an offer from top ranked worker, it is optimal to make an offer to her.

The only firm strategic decision is when it receives a signal from second top worker. We call firm strategy “respond” if it makes an offer to its second top worker in this case and “ignore” if it makes an offer to top ranked worker. Then, there could be two symmetric equilibria when both firms either “ignore” or “respond”. We have the following characterization of equilibria existence and welfare for different assumptions on firm utilities in the offer game with signals.

	$u(2) \leq \frac{1}{2}u(1)$	$u(2) > \frac{1}{2}u(1)$
<i>(ignore, ignore)</i>	$\frac{3}{4}u(1)$, equilibrium	-, not equilibrium
<i>(respond, respond)</i>	$\frac{5}{8}u(1) + \frac{1}{4}u(2)$, equilibrium	$\frac{5}{8}u(1) + \frac{1}{4}u(2)$, equilibrium

If firm utilities satisfy $u(2) \leq \frac{1}{2}u(1)$ there exist two equilibria in the offer game with signals. One of them coincides with the equilibrium in the game with no signals. The firm welfare is weakly smaller in the offer game with signals compare to the offer game with no signals¹⁵. However, if firm utilities satisfy $u(2) > \frac{1}{2}u(1)$, there exists only one equilibrium in the offer game with signals when firms always respond to signals. In this case, firm welfare in the unique equilibrium in the offer game with signals is greater than in the unique equilibrium in the game with no signals. \square

¹⁵See Section 5 for welfare results in markets with one block of firms.

B Appendix. Equilibrium Ranking in a Single Block

Proof of Theorem 4.

We show below that the subgame when firms choose their cutoff vectors is a game with strategic complementarities of Milgrom and Roberts (1990). In order to show it, we first define a partial order of cutoff strategies. The strategy of firm f is a vector of cutoffs, $s_f = (j_f^1, \dots, j_f^W)$. Each cutoff $j_f^k \in \{1, \dots, W\}$ corresponds to a positive number of signals firm f could receive. Using notation of the proof of Theorem 2, we denote the set of firms cutoff strategies as S with $s = (s_1, \dots, s_F)$ being its typical element. We impose the following partial order on S , $s \geq_S s' \Leftrightarrow s_f \geq s'_f \Leftrightarrow j_f^k \geq (j'_f)^k$ for any $f \in \{1, \dots, F\}$ and $k \in \{1, \dots, W\}$. This partial order is reflexive, antisymmetric, and transitive.

Now in order to show that the subgame is a game with strategic complementarities, we need to check whether $E_\theta(\pi_f(s_f, s_{-f}, \theta))$ is supermodular in s_f and whether $E_\theta(\pi_f(s_f, s_{-f}, \theta))$ has increasing differences in s_f and s_{-f} . The former one follows from the fact that a change of one cutoff does not influence the payoff from the change of the other cutoff. Namely, if we consider $s_f^1 = (\dots, j_l, \dots, j_k, \dots)$, $s_f^2 = (\dots, j'_l, \dots, j_k, \dots)$, $s_f^3 = (\dots, j_l, \dots, j'_k, \dots)$ and $s_f^4 = (\dots, j'_l, \dots, j'_k, \dots)$ for some $l, k \in \{1, \dots, W\}$, then

$$E_\theta(\pi_f(s_f^1, s_{-f}, \theta)) - E_\theta(\pi_f(s_f^2, s_{-f}, \theta)) = E_\theta(\pi_f(s_f^3, s_{-f}, \theta)) - E_\theta(\pi_f(s_f^4, s_{-f}, \theta))$$

The fact that $E_\theta(\pi_f(s_f, s_{-f}, \theta))$ has increasing differences in s_f and s_{-f} follows from Proposition 4. Namely, for any s_f, s_{-f}, s'_f , and s'_{-f} such that $s'_f \geq s_f$ and $s'_{-f} \geq s_{-f}$ we have that

$$E_\theta(\pi_f(s'_f, s'_{-f}, \theta)) - E_\theta(\pi_f(s_f, s'_{-f}, \theta)) \geq E_\theta(\pi_f(s'_f, s_{-f}, \theta)) - E_\theta(\pi_f(s_f, s_{-f}, \theta))$$

As a consequence, we have that the subgame when firms choose their cutoff vectors is a game with strategic complementarities. Therefore, we can apply Theorem 5 of Milgrom and Roberts (1990) that establishes the existence of largest and smallest pure strategy equilibrium. \square

Proof of Proposition 7.

Firm f strategy s_f differ from s'_f only in that s'_f has weakly greater cutoffs (responds more to signals). Let us consider some firm $f' \in -f$. For each profile of preferences $\theta_{f'}$ and a set of signals h , firm f' either makes an offer to $TSW_{f'}(\theta_{f'}, h)$ or $TRW_{f'}(\theta_{f'}, h)$. Since, workers send their signals to their best firms (there is only one block), the change in firm f strategy does not influence the probability $TSW_{f'}$ accepts firm f' offer. However, as we have showed in the proof of Proposition 4, the probability that $TRW_{f'}$ accepts firm f' offer decreases. Overall, we get a negative spillover effect that if firm f responds more to signals,

the expected payoff of firm $f' \in -f$ decreases.

$$E_{\theta}(\pi_{f'}(s_f, s_{-f}, \theta)) \geq E_{\theta}(\pi_{f'}(s'_f, s_{-f}, \theta)).$$

□

Proof of Theorem 5.

The result that the expected number of matches and the expected welfare of workers is higher in the equilibrium with higher cutoffs is an immediate consequence of Proposition 5 and Proposition 6. In order to show that firms have lower expected payoffs in such equilibrium we use result of Proposition 7. Let us consider two symmetric equilibria where firms play s and s' cutoffs, such that $s, s' \in \{1, \dots, W\}^W$ and $s' \geq s$. From the definition of an equilibrium strategy we have:

$$E_{\theta}[\pi_f(s_f, s_{-f}, \theta)] \geq E_{\theta}[\pi_f(s'_f, s_{-f}, \theta)]$$

Using the result of Proposition 7 we may proceed with

$$E_{\theta}[\pi_f(s'_f, s_{-f}, \theta)] \geq E_{\theta}[\pi_f(s'_f, s'_{-f}, \theta)]$$

Therefore

$$E_{\theta}[\pi_f(s_f, s_{-f}, \theta)] \geq E_{\theta}[\pi_f(s'_f, s'_{-f}, \theta)]$$

□

C Appendix. Market Structure and The Value of a Signaling Mechanism.

Proposition C1 *Under the assumption that*

$$u(W) > \frac{W}{F} \left(1 - \left(1 - \frac{1}{W}\right)^F\right) u(1) \quad (4)$$

there is the unique equilibrium that satisfies criterion D1 of Cho and Kreps in the offer game with signals. Each worker sends a signal to top preferred firms. Each firm f makes an offer to TSW_f if it receives at least one signal; otherwise, it makes an offer to TRW_f .

Proof .

Let us show that under condition 4 even if TSW_f is the last worker in firm f preference list, firm f still optimally makes her an offer.

Proposition 4 shows that if firms $-f$ respond more to signals, i.e. increase their cutoffs, it is also optimal for firm f to respond more to signals. Therefore, if firm f responds to signals when no other firm does, it will optimally respond to signals when other firm respond. Hence, it is enough to consider the incentives of firm f when firms $-f$ do not respond to signals and always make an offer their top ranked workers.

Let us consider some realized profile of preferences of firm f and denote TRW_f as w . If firms $-f$ does not respond to signals, then some firm among $-f$ makes an offer to worker w with probability $p = \frac{1}{W}$. Therefore, the probability that the offer of firm f to worker w is accepted equals:

$$(1-p)^{F-1} + \dots C_{F-1}^j p^j (1-p)^{F-1-j} \frac{1}{j+1} + \dots + p^{F-1} \frac{1}{F} \quad (5)$$

Intuitively, j firms among other $F-1$ firms simultaneously make an offer to worker w with probability $C_{F-1}^j p^j (1-p)^{F-1-j}$. Therefore, firm f is matched with worker w only with probability $\frac{1}{j+1}$ because worker w preferences are uniformly distributed. The sum over all possible j from 0 to $F-1$ gives us the overall probability of firm f offer being accepted. We

can simplify this expression as

$$\sum_{j=0}^{F-1} C_{F-1}^j p^j (1-p)^{F-1-j} \frac{1}{j+1} \quad (6)$$

$$= \sum_{j=0}^{F-1} \frac{(F-1)!}{j!(F-1-j)!} p^j (1-p)^{F-1-j} \frac{1}{j+1} \quad (7)$$

$$= \sum_{j=0}^{F-1} \frac{1}{Fp} \frac{F!}{(j+1)!(F-(1+j))!} p^{j+1} (1-p)^{F-(1+j)} \quad (8)$$

$$= \frac{1}{Fp} \sum_{j=0}^{F-1} \frac{F!}{t!(F-t)!} p^t (1-p) \quad (9)$$

$$= \frac{1}{Fp} \left(\sum_{j=0}^{F-1} \frac{F!}{t!(F-t)!} p^t (1-p)^{F-t} - (1-p)^F \right) \quad (10)$$

$$= \frac{1}{Fp} \left(1 - (1-p)^F \right) = \frac{W}{F} \left(1 - \left(1 - \frac{1}{W} \right)^F \right) \quad (11)$$

Alternatively, if firm f makes an offer to its top signaled worker, its offer is accepted with probability equal one. Therefore, it is optimal for firm to make an offer to its last ranked if and only if $u(W) > \frac{W}{F} \left(1 - \left(1 - \frac{1}{W} \right)^F \right) u(1)$. Using the discussion above, we conclude that under assumption 4 there is no other non-babbling symmetric equilibrium in the offer game with signals. \square

Proof of Proposition 8.

We first calculate an explicit formula for the increase in the expected number of matches from the introduction of the signaling mechanism with one signal.

Lemma C1 *Let us consider a market with W workers and F firms. The expected number of matches in the offer game with no signals equals*

$$U(F, W) = W \left(1 - \left(1 - \frac{1}{W} \right)^F \right) \quad (12)$$

The expected number of matches in the offer game with signals when each worker can send up to one signal equals

$$S(F, W) = F \left(1 - \left(\frac{F-1}{F} \right)^W + \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} \left(1 - \frac{F-1}{W} \left(1 - \left(\frac{F-2}{F-1} \right)^W \right) \right) * \right. \\ \left. * \left(1 - \left(1 - \frac{1}{W} \left(\frac{F-2}{F-1} \right)^{W-1} \right)^{F-1} \right) \right) \quad (13)$$

Proof of Lemma C1.

Let us first calculate the expected number of matches in the offer game with no signals. The unique symmetric non-babbling equilibrium when agents use anonymous strategies is

the following. Each firm makes an offer to its top ranked worker and each worker accepts the best offer among available ones. Proposition C1 calculates the probability of firm f being matched to its top ranked worker, which equals

$$\frac{W}{F} \left(1 - \left(1 - \frac{1}{W} \right)^F \right)$$

Therefore, the expected total number of matches in the game with no signals equals

$$U(F, W) = W \left(1 - \left(1 - \frac{1}{W} \right)^F \right) \quad (14)$$

Let us now calculate the expected number of matches in the offer game with signals, when each worker can send only one signal. There is the unique symmetric non-babbling equilibrium in the offer game with signals. Each worker sends her signal to her top firm and each firm makes its offer to top signaled worker (TSW) if it receives at least one signal, otherwise it makes an offer to its TRW .

We first calculate ex-ante probability of being matched by some firm f . We denote the set of workers that send signal to firm f as $h_f \subset W$. If firm f receives at least one signal, $|h_f| > 0$, it guarantees itself a match, because each worker send her signal to her top firm. If firm f receives no signals, it makes an offer to its top ranked worker. This worker accepts firm f offer only if firm f is the best firm in the worker's preferences among the firms she receives an offer from. Let us denote the probability that TRW_f accepts firm f 's offer under the condition that firm f receives no signals as

$$P_{TRW_f, |h_f|=0} = P(TRW_f \text{ accepts firm } f\text{'s offer} \mid |h_f| = 0)$$

Then ex-ante probability that firm f is matched equals

$$S^f(F, W) = P(|h_f| > 0) * 1 + P(|h_f| = 0) * P_{TRW_f, |h_f|=0} \quad (15)$$

If firm f receives no signals, $P(|h_f| = 0)$, it makes an offer to TRW_f that we denote as worker $w = TRW_f$. Worker w receives an offer from its top ranked firm, say firm f_0 , conditional on firm f receiving no signals, $|h_f| = 0$, with probability equal

$$\begin{aligned} G &= P(|h_{f_0}| = 1 \mid |h_f| = 0) * 1 + \dots + P(|h_{f_0}| = W \mid |h_f| = 0) * \frac{1}{W} \\ &= \sum_{j=0}^{W-1} C_{W-1}^j \left(\frac{1}{F-1} \right)^j \left(1 - \frac{1}{F-1} \right)^{W-j-1} \frac{1}{j+1} \end{aligned} \quad (16)$$

Intuitively, firm f_0 receives a signal from a particular worker with probability $\frac{1}{F-1}$ (note that firm f receives no signals). Then, if firm f_0 receives signals from j other workers, worker w receives an offer from firm f_0 with probability $\frac{1}{j+1}$. Similarly to equation (6) the expression

for G simplifies to

$$G = \frac{F-1}{W} \left(1 - \left(1 - \frac{1}{F-1}\right)^W\right)$$

Firm f can be matched with worker w only if worker w does not receive an offer from its top firm, which happens with probability $1 - G$. If worker w does not receive an offer from her top firm – firm f_0 – firm f competes with other firms that have received no signals from workers. The probability that some firm f' among firms $F \setminus \{f, f_0\}$ receives no signals conditional on the fact that worker w sends her signal to firm f_0 and firm f receives no signals $|h_f| = 0$ equals $r = \left(1 - \frac{1}{F-1}\right)^{W-1}$. Note that the probability that firm f' does not receive a signal from a worker equals $1 - \frac{1}{F-1}$, because firm f receives no signals. There are also only $W - 1$ workers that can send a signal to firm f' , because worker w sends her signal to firm f_0 .

Therefore, the probability that some firm f' among firms $F \setminus \{f, f_0\}$ receives no signals and makes an offer to worker w , conditional on the fact that worker w sends her signal to firm f_0 , equals $\frac{r}{W}$. Therefore, the probability that worker w prefers offer of firm f to other offers (conditional on the fact that firm f receives no signals and worker w sends her signal to firm f_0) equals¹⁶

$$\sum_{j=0}^{F-2} C_{F-2}^j \left(\frac{r}{W}\right)^j \left(1 - \frac{r}{W}\right)^{F-2-j} \frac{1}{j+1} = \frac{W}{(F-1)r} \left(1 - \left(1 - \frac{r}{W}\right)^{F-1}\right) \quad (17)$$

Then the probability that worker w accepts firm f 's offer equals

$$P_{TRWf, |h_f|=0} = (1 - G) \left(\frac{W}{(F-1)r} \left(1 - \left(1 - \frac{r}{W}\right)^{F-1}\right)\right)$$

Taking into account that firm f receives no signals with probability equal

$$P(|h_f| = 0) = \left(1 - \frac{1}{F}\right)^W$$

the probability of firm f being matched in the offer game with signals equals

$$\begin{aligned} S^f(F, W) &= 1 - \left(1 - \frac{1}{F}\right)^W + \left(1 - \frac{1}{F}\right)^W * P_{TRWf, |h_f|=0} \\ &= 1 - \left(1 - \frac{1}{F}\right)^W + \left(1 - \frac{1}{F}\right)^W \frac{W}{(F-1)r} \left(1 - \frac{F-1}{W} \left(1 - \left(1 - \frac{1}{F-1}\right)^W\right)\right) * \\ &\quad * \left(1 - \left(1 - \frac{r}{W}\right)^{F-1}\right) \end{aligned} \quad (18)$$

where $r = \left(1 - \frac{1}{F-1}\right)^{W-1}$. The expected total number of matches in the offer game with signals equals $S(F, W) = FS^f(F, W)$. \square

¹⁶Note, that the maximum number of offers worker w could get equals to $M - 1$ as it does not receive an offer from its top firm f_0 .

Lemma C1 establishes the expected total number of matches in the offer game with and with no signals. Let us first fix W and calculate where the increase in the expected number of matches from the introduction of the signaling mechanism, $D(F, W) = S(F, W) - U(F, W)$, attains its maximum. In order to obtain the result of the proposition we consider large markets, where the number of firms and the number of workers are large, and we use Taylor's expansion formula

$$(1 - a)^b = \exp(-ab + O(a^2b)) \quad (19)$$

where $O(a^2b)$ is a function that is smaller than a constant for large values of a^2b . If we denote $x = \frac{F}{W}$ the expected number of matches in the offer game with no signals can be approximates as

$$U(F, W) = W \left(1 - \left(1 - \frac{1}{W} \right)^F \right) = W(1 - e^{-x+O(x/W)})$$

Let us now consider the expected number of matches in the offer game with signals. Using the result of lemma C1 we get

$$S(F, W) = Wx \left(1 - e^{-1/x+O(1/(x^2W))} + A * B \right)$$

where

$$\begin{aligned} A &= \left(1 - \frac{F-1}{W} \left(1 - \left(\frac{F-2}{F-1} \right)^W \right) \right) \\ B &= \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} \left(1 - \left(1 - \frac{1}{w} \left(\frac{F-2}{F-1} \right)^{W-1} \right)^{F-1} \right) \end{aligned}$$

We first calculate an approximation of A for large markets. Using (19) we get that

$$1 - \left(1 - \frac{1}{F-1} \right)^W = 1 - e^{-x+O(1/(x^2W))}$$

and

$$A = 1 - x \left(1 - e^{-1/x+O(1/(x^2W))} \right) + O(1/(xW))$$

We now calculate an approximation of B for large markets.

$$\begin{aligned} \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}} &= \frac{W}{F} \left(\frac{F-1}{F} \right)^{W-1} \left(\frac{F-1}{F-2} \right)^{W-1} \\ &= \frac{1}{x} e^{-(W-1)/F+O(1/(x^2W))} e^{(W-1)/(F-1)+O(1/(x^2W))} \\ &= \frac{1}{x} e^{O(1/(x^2W))} \end{aligned}$$

Also, we have that

$$\begin{aligned} \left(1 - \left(1 - \frac{Z}{W}\right)^{F-1}\right) &= 1 - e^{-Z(F-1)/W + O(x/W)} \\ &= 1 - e^{-Zx + O(x/W)} \end{aligned}$$

where $Z = \left(\frac{F-2}{F-1}\right)^{W-1} = e^{-1/x + O(1/(x^2W))}$. Then, we have that

$$\begin{aligned} B &= \frac{W(F-1)^{2W-2}}{FW(F-2)^{W-1}} * \left(1 - \left(1 - \frac{1}{W}\left(\frac{F-2}{F-1}\right)^{W-1}\right)^{F-1}\right) \\ &= \frac{1}{x} e^{O(1/(x^2W))} (1 - e^{-xe^{-1/x} + O(x/W)}) \end{aligned}$$

Overall, we have

$$\begin{aligned} D(F, W) &= Wx \left(1 - e^{-1/x + O(1/(x^2W))} + \left(1 - x \left(1 - e^{-1/x + O(1/(x^2W))}\right) + O(1/(xW))\right) * \right. \\ &\quad \left. * \frac{1}{x} e^{O(1/(x^2W))} (1 - e^{-xe^{-1/x} + O(x/W)}) \right) - \\ &\quad - W(1 - e^{-x + O(x/W)}) \\ &= W \left(x - xe^{-1/x} + (1 - x(1 - e^{-1/x})) (1 - e^{-xe^{-1/x}}) - 1 + e^{-x} \right) + O(1) \\ &= W\alpha(x) + O(1) \end{aligned}$$

where $\alpha(x)$ is a positive quasi-concave function that attains maximum at $x_0 \simeq 1.012113$. Therefore, for fixed W , $D(F, W)$ attains its maximum value at $F = x_0W + O(1)$.

Similar to the previous derivation, we fix F and calculate the value of W where $D(F, W)$ attains its maximum.

$$\begin{aligned} D(F, W) &= F \left(1 - e^{-1/x + O(1/(x^2W))} + \left(1 - x \left(1 - e^{-1/x + O(1/(x^2W))}\right) + O(1/(xW))\right) \right. \\ &\quad \left. * \frac{1}{x} e^{O(1/(x^2W))} (1 - e^{-xe^{-1/x} + O(x/W)}) \right) \\ &\quad - \frac{F}{x} (1 - e^{-x + O(x^2/F)}) \\ &= F \left(1 - e^{-1/x} + (1 - x(1 - e^{-1/x})) \frac{1}{x} (1 - e^{-xe^{-1/x}}) - \frac{1}{x} (1 - e^{-x}) \right) + O(1) \\ &= F\beta(x) + O(1) \end{aligned}$$

where $\beta(x)$ is a positive quasi-concave function that attains maximum at $x_{00} \simeq 0.53074$. Therefore, for fixed F , $D(F, W)$ attains its maximum value at $W = y_0F + O(1)$, where $y_0 = 1/x_{00} = 1.8842$. \square

Proposition C2 (K signals and I interviews) 1) Consider a market when each worker has I interview positions to fill. There is the unique equilibrium of the pure coordination model with no signals. Each firm f makes an offer to TRW_f and each worker accepts the best I offers from those it receives.

2) In addition, assume that if each worker can send up to K signals and

$$H(K, W, I)u(W) > H(F, W, I)u(1),$$

where

$$H(F, W, I) = \sum_{j=0}^{I-1} C_{F-1}^j \left(\frac{1}{W}\right)^j \left(1 - \frac{1}{W}\right)^{F-1-j} + \sum_{j=I}^{F-1} C_{F-1}^j \left(\frac{1}{W}\right)^j \left(1 - \frac{1}{W}\right)^{F-1-j} \frac{I}{j+1}$$

Then, there is also the unique symmetric non-babbling equilibrium that satisfies criterion D1 of Cho and Kreps in the offer game with signals. Each worker send K signals to her most preferred K firms and accepts the best I offers among those available. Each firm f makes an offer to TSW_f if it receives at least one signal; otherwise it makes an offer to TRW_f .

Proof.

Since workers make decision only on the last stage the first statement about the offer game with no signals follows directly from Theorem 1.

Let us now consider the pure coordination game with signals. If firms employ symmetric strategies, then the probability of a worker w receiving an offer from a firm conditional on w sending a signal to it is the same across all firms. Similarly, the probability of w receiving an offer from a firm conditional on w not sending a signal to it is the same across all firms. We denote the former one as q and the latter one as p . The argument similar to the one in Proposition 1 show that $q \geq p$. Therefore, it is optimal for each worker to send K signals to her top firms in any non-babbling symmetric equilibrium, i.e. $q > p$.

Since, workers send identical signals, a firm f prefers to make an offer to TSW_f rather to any other worker who sends a signal to it. Similar, a firm f prefers to make an offer to TRW_f rather than to any other worker who do not send a signal to it (similar to Proposition 2). As one could see the statements of Proposition 2 and Proposition 3 also hold. The proofs of these propositions for the case of K signals and I interviews repeat the argument of the case of one signal and one interview.

We now use logic of Proposition C1 to establish conditions on utility function such that it is optimal for a firm to make an offer to TSW_f even if she is the last ranked worker.

Let us consider some realized profile of preferences of firm f and denote TRW_f as w . If firms $-f$ does not respond to signals, then some firm makes its offer to worker w with probability $p = \frac{1}{W}$. Therefore, the probability that firm f offer to worker w is accepted equals:

$$H(F, W, I) = \sum_{j=0}^{I-1} C_{F-1}^j \left(\frac{1}{W}\right)^j \left(1 - \frac{1}{W}\right)^{F-1-j} + \sum_{j=I}^{F-1} C_{F-1}^j \left(\frac{1}{W}\right)^j \left(1 - \frac{1}{W}\right)^{F-1-j} \frac{I}{j+1}$$

If there are fewer than $I - 1$ firms among $-f$ make an offer to worker w , firm f is matched with unit probability. Otherwise firm f is matched with worker w only with probability $\frac{I}{j+1}$. Unfortunately, this expression cannot be simplified as in Proposition C1. However, if firm f makes an offer to TSW_f , it only know that it is among K top firms. Therefore, the probability that worker TSW_f accepts firm f offer equals $H(K, W, I)$. Therefore, if $H(K, W, I)u(W) > H(F, W, I)u(1)$, it is always optimal for firm f to make an offer to TSW_f . \square

Proposition C3 *Under the assumptions*

$$\begin{aligned} u(W) &> \frac{W}{F} \left(1 - \left(1 - \frac{1}{W}\right)^F\right) u(1) \\ u(W) &> \delta u(1), \quad v(W) > \delta v(1) \end{aligned}$$

1. *There is the unique symmetric sequential equilibrium in the offer game with no signals and L periods of interaction: each firm makes an offer to its most preferred worker and each worker accepts its best offer in each period.*
2. *There is the unique symmetric sequential non-babbling equilibrium in the offer game with signals and L periods of interaction that satisfies criterion D1 of Cho and Kreps: each worker sends her signal to her top firm at period 0. Each worker accepts the best available offer at each period. Each firm makes an offer at period $l = 1, \dots, L$ to its TSW among workers who are in the market; otherwise the firm makes an offer to its TRW among workers who are in the market.*

Proof.

Let us first consider the offer game with no signals and L periods of interaction. Using backward induction we consider the last stage of the game. Since, the last stage of the game is one period offer game with no signals. The only symmetric equilibrium in this subgame: each firm makes an offer to its top ranked worker and each worker accepts best available offer.

Assumptions $u(W) > \delta u(1)$ and $v(W) > \delta v(1)$ guarantee that there is no incentives to hold offers or make dynamically strategic offers. Since firms $-f$ use symmetric anonymous strategies at stage $L - 1$ and stage L , the only optimal strategy of firm f at stage $L - 1$ is to make its offer to its top ranked offer. Each worker, who receives at least one offer at stage $L - 1$, optimally accepts the best available offer immediately. Similar logic applies to the other stages.

Let us now consider the offer game with signals and L periods of interaction. The symmetry of workers $-w$ strategies and anonymous of firms strategies guarantee that the

probability that each firm makes an offer to worker w (considering all L periods), conditionally on receiving her signal/not receiving her signal is the same across firms. Therefore, workers should still optimally send their signals to top firms in the offer game with signals and L periods of interaction.

In addition, there is only first stage of the game when signals play role. Since, $u(W) > \delta u(1)$ and $u(W) > \frac{W}{F} \left(1 - \left(1 - \frac{1}{W}\right)^F\right) u(1)$ each firm makes an offer to its top signaled worker if it receives at least one signals at stage 1. Since $v(W) > \delta v(1)$ workers accepts best available offers at stage 1. Now if you consider stage 2, there are only firms that receive no signal at stage 1 and whose offer is not accepted at stage 1. Therefore, the logic of backward induction of statement 1 applies to period 2 through period L in the offer game with signals.

□

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