Lecture Notes for Alvarez and Lucas (2006)

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- L consumers/workers - inelastic supply, no capital
- Single (non-tradable) final consumption good, quantity per capita is c, price p
- Value of total consumption (GDP) is Lpc
- Continuum of tradable inputs in $[0, 1]$ used to produce a composite input via a CES with $\eta > 1$,

$$q = \left[\int_0^1 q(u)^{1-1/\eta} du \right]^{\eta/(\eta-1)}$$

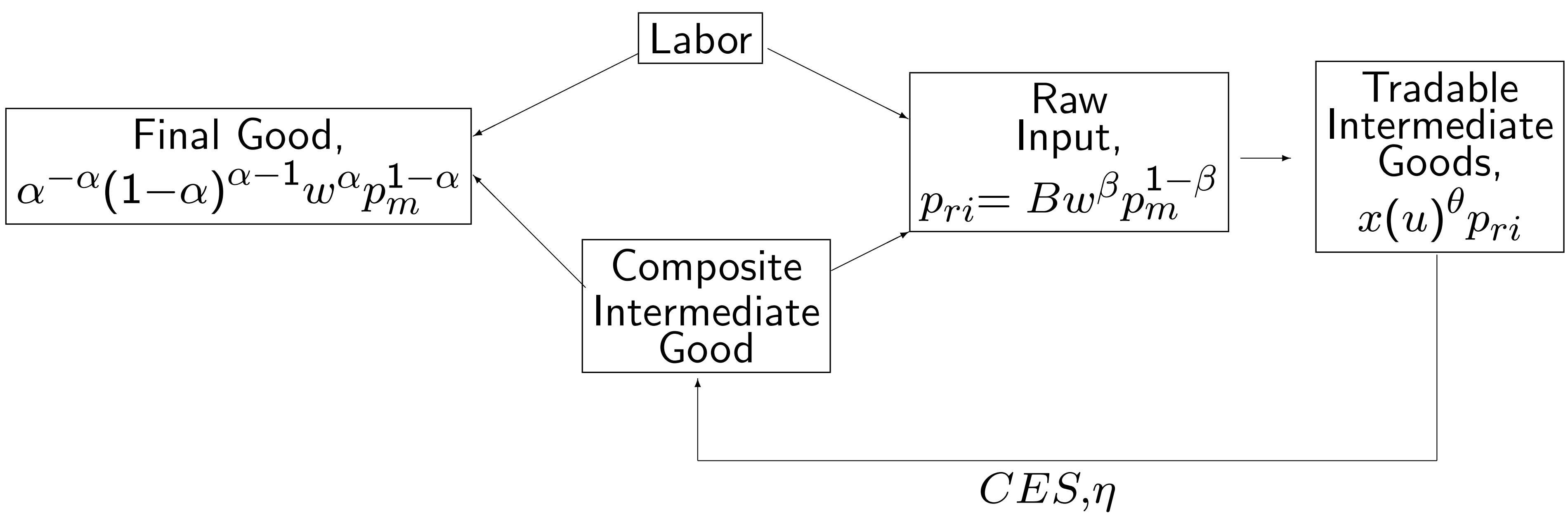
- This composite input together with labor is used to produce the final good according to a CD production function with labor share α,

$$c = s_f^{\alpha} q_f^{1-\alpha}$$

- Inputs are produced with a CD production function with labor and the composite input,

$$q(u) = x(u)^{-\theta} s(u)^{\beta} q_m(u)^{1-\beta}$$

Figure 1: The Production Structure



- Here x is a cost parameter, iid across u with $x \sim \exp(\lambda)$

- Instead of u, we keep track of goods by x,

$$
\begin{aligned}
q &= \left[\lambda \int_0^\infty e^{-\lambda x} q(x)^{1-1/\eta} dx\right]^{\eta/(\eta-1)} \\
q(x) &= x^{-\theta} s(x)^\beta q_m(x)^{1-\beta}
\end{aligned}
$$

- Resource constraints:

$$
\begin{aligned}
s_f + s_m &= s_f + \lambda \int_0^\infty e^{-\lambda x} s(x) dx = 1 \\
q_f + q_m &= q_f + \lambda \int_0^\infty e^{-\lambda x} q(x) dx = q
\end{aligned}
$$

- Cost minimization for c, q, and $q(x)$ implies

$$p_m^{1-\eta} = \lambda \int_0^{\infty} e^{-\lambda x} p(x)^{1-\eta} dx \;\; \text{with} \;\; q(x) = \left(\frac{p(x)}{p_m} \right)^{-\eta} q \tag{1}$$

$$p = f(w, p_m) \equiv \alpha^{-\alpha} (1-\alpha)^{-(1-\alpha)} w^{\alpha} p_m^{1-\alpha} \tag{2}$$

$$p(x) = g(w, p_m) \equiv B x^{\theta} w^{\beta} p_m^{1-\beta} \tag{3}$$

- For future reference, note that from (2) can get

$$\frac{s_f}{q_f} = \frac{f_1}{f_2} = \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{p_m}{w} \right) \tag{4}$$

whereas from (3) can get

$$\frac{s(x)}{q(x)} = \frac{g_1}{g_2} = \left(\frac{\beta}{1-\beta} \right) \left(\frac{p_m}{w} \right) \tag{5}$$

- Plugging (3) into (1), using change of variable $z = \lambda x$, and solving for p_m in terms of w yields

$$p_m = (AB)^{1/\beta} \lambda^{-\theta/\beta} w \tag{6}$$

where

$$A \equiv \Gamma(1 + \theta(1 - \eta))^{1/(1-\eta)} \quad \text{and} \quad 1 + \theta(1 - \eta) > 0$$

- Plugging this into (2) yields

$$p = \alpha^{-\alpha}(1 - \alpha)^{-(1-\alpha)}(AB)^{(1-\alpha)/\beta} \lambda^{-\theta(1-\alpha)/\beta} w \tag{7}$$

and into (3) yields

$$p(x) = A^{(1-\beta)/\beta} B^{1/\beta} x^{\theta} \lambda^{-\theta(1-\beta)/\beta} w \tag{8}$$

- Now we have p_m, $p(x)$, and p as multiples of w. This completes determination of prices.

- Note that $c = w/p = \alpha^\alpha (1-\alpha)^{(1-\alpha)} (AB)^{-(1-\alpha)/\beta} \lambda^{\theta(1-\alpha)/\beta}$

- Now determine quantities: allocation of labor to the final good and inputs, and same for q. The value of production of the composite good is equal to spending on labor (ws_m) and inputs ($p_m q_m$):

$$\begin{aligned} ws_m + p_m q_m &= p_m q \\ w(1-s_f) + p_m q_m &= q_f p_m + q_m p_m \\ w(1-s_f) &= q_f p_m \end{aligned}$$

and hence

$$\frac{1-s_f}{q_f} = \frac{p_m}{w} \tag{9}$$

- Using this together with (4) (i.e., $\frac{s_f}{q_f} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{p_m}{w}\right)$) yields

$$\frac{s_f}{1 - s_f} = \frac{\alpha}{1 - \alpha}$$

and hence,

$$s_f = \alpha$$

From (9) then

$$q_f = (1 - \alpha)(AB)^{-1/\beta}\lambda^{\theta/\beta}$$

whereas from (5) get

$$q_m = (1 - \alpha)\left(\frac{1 - \beta}{\beta}\right)(AB)^{-1/\beta}\lambda^{\theta/\beta}$$

- Now we allow for international trade

- n countries indexed by i, possibly differing in λ_i, L_i

- Notation: $L = (L_1, ..., L_n)$, $\lambda = (\lambda_1, ..., \lambda_n)$, $w = (w_1, ..., w_n)$

- Transportation costs: a unit of a good sent j to i results in $k_{ij} \leq 1$ arriving in i

- Thus, $k_{ij} = 1/d_{ij}$. Also, if unit production cost of a good is c in country j then the cost of a unit in country i would be c/k_{ij} so that producer receives c.

- Assumptions: $k_{ii} = 1$, symmetry ($k_{ij} = k_{ji}$), and the triangular inequality,

$$k_{ij} \geq k_{ik} k_{kj} \quad \text{for all } i, j, k$$

- Tariffs: ω_{ij} is the fraction of each dollar spent in i on goods made in j that arrives as payment to a seller in j

- Note that this affects costs in the same way as transportation costs: if unit production cost of a good is c in country j then the price of that good in country i would be $c/k_{ij}\omega_{ij}$ so that producer receives c.

- How does ω relate to the ad-valorem tariff τ? If exporter charges p then domestic consumers pay $(1 + \tau)c/k = c/k\omega$, hence

$$1 + \tau = 1/\omega$$

- Now we keep track of goods as $x = (x_1, ..., x_n)$, and

$$p_i(x) = B \min_j \left\{ x_j^{\theta} \frac{w_j^{\beta} p_{mj}^{1-\beta}}{k_{ij}\omega_{ij}} \right\}$$

- Letting $\phi(x)$ be the joint density of x, then

$$q_i = \left[\int_{\mathbb{R}+} q_i(x)^{1-1/\eta} \phi(x) dx \right]^{\eta/(\eta-1)}$$

and

$$p_{mi}^{1-\eta} = \int_{\mathbb{R}+} p_i(x)^{1-1/\eta} \phi(x) dx$$

- Let

$$z^\theta \equiv \min_j \left\{ x_j^\theta \frac{w_j^\beta p_{mj}^{1-\beta}}{k_{ij}\omega_{ij}} \right\}$$

- Note that (1) if $x \sim \exp(\lambda)$ and $a > 0$ then $ax \sim \exp(\lambda/a)$, and (2) if x_1 and x_2 are independent, and $x_i \sim \exp(\lambda_i)$ then $\min\{x_i\} \sim \exp(\lambda_1 + \lambda_2)$

- This implies that $z \sim \exp(\psi_i)$, where

$$\psi_i = \sum_j \psi_{ij} \quad \text{and} \quad \psi_{ij} = \left(\frac{w_j^\beta p_{mj}^{1-\beta}}{k_{ij}\omega_{ij}} \right)^{-1/\theta} \lambda_j$$

- Leaving aside the i subscript for now, this implies that

$$p_m^{1-\eta} = \psi \int (Bz^\theta)^{1-\eta} e^{-\psi z} dz$$

- Letting $\mu \equiv B^{-1/\theta}\psi$ and using $\upsilon = (\psi/\mu)z$ then

$$Bz^\theta = B\,(\upsilon\mu/\upsilon)^\theta = (\mu/\psi)^{-\theta}\,(\upsilon\mu/\psi)^\theta = \upsilon^\theta$$

and hence $p_m^{1-\eta} = \mu \int \upsilon^{\theta(1-\eta)} e^{-\mu\upsilon} d\upsilon$. Using $t = \mu\upsilon$ and A from above then $p_{mi}^{1-\eta} = \mu^{-\theta(1-\mu)} A^{1-\eta}$, so

$$p_{mi} = AB \left(\sum_j \psi_{ij} \right)^{-\theta}$$

- We have

$$p_{mi} = AB\left(\sum_j \left(\frac{w_j^{\beta} {p_{mj}}^{1-\beta}}{k_{ij}\omega_{ij}}\right)^{-1/\theta} \lambda_j\right)^{-\theta} \tag{10}$$

- This is a system of n equations in prices $p_{m1}, ..., p_{mn}$, giving us an implicit function $p_m(w)$, analogous to the closed economy result that $p_m(w) = (AB)^{1/\beta}\lambda^{-\theta/\beta}w$.

- We can check that this is what we would get by letting $k_{ij}w_{ij} \to 0$ for all $i \neq j$, which yields

$$\begin{aligned} p_{mi} &= AB\left(\left(w_i^{\beta} {p_{mi}}^{1-\beta}\right)^{-1/\theta} \lambda_i\right)^{-\theta} \\ &\to p_{mi}^{\beta} = \left(AB\lambda_i^{-\theta}\right)^{1/\beta} w_i \end{aligned}$$

- The next step is to obtain a system of equations for $w = (w_1, ...w_n)$. Here we use the trade-balance conditions.

- Let D_{ij} be the share of expenditure on tradables in country i that is spent on goods from j. From EK-kind analysis we know that

$$D_{ij} = \frac{\psi_{ij}}{\sum_k \psi_{ik}}$$

- Using the definition of ψ_{ij} and (10) then

$$D_{ij}(w) = \frac{\left(\frac{w_j^\beta p_{mj}^{1-\beta}}{k_{ij}\omega_{ij}}\right)^{-1/\theta} \lambda_j}{\sum_k \left(\frac{w_k^\beta p_{mk}^{1-\beta}}{k_{ik}\omega_{ik}}\right)^{-1/\theta} \lambda_k} = (AB)^{-1/\theta} \left(\frac{w_j^\beta p_{mj}(w)^{1-\beta}}{p_{mi}k_{ij}\omega_{ij}}\right)^{-1/\theta} \lambda_j$$

- Firms in country i spend $L_i p_{mi} q_i$ on tradables, but the country spends $L_i p_{mi} q_i \sum_j D_{ij} \omega_{ij}$, whereas it receives $\sum_j L_j p_{mj} q_j D_{ji} \omega_{ji}$, hence trade blance is

$$L_i p_{mi}(w) q_i F_i(w) = \sum_j L_j p_{mj}(w) q_j D_{ji}(w) \omega_{ji} \tag{11}$$

where

$$F_i(w) \equiv \sum_j D_{ij}(w) \omega_{ij}$$

is an average tariff.

- This is a system of equations in w and $q_1, ..., q_n$. Now we need to solve for the $q's$ in terms of w.

- We have the following two key equations from the demand for labor and the composite intermediate good in the production of final and intermediate goods:

$$\frac{s_{fi}}{q_{fi}} = \left(\frac{\alpha}{1-\alpha}\right)\frac{p_{mi}}{w_i} \tag{12}$$

and

$$\frac{s_{mi}}{q_{mi}} = \left(\frac{\beta}{1-\beta}\right)\frac{p_{mi}}{w_i} \tag{13}$$

Moreover, given that total cost of tradeables must equal the value of those tradeables, then

$$L_i w_i s_{mi} + L p_{mi} q_{mi} = \sum\nolimits_j L_j p_{mj} q_j D_{ji} \omega_{ji} \tag{14}$$

But using the trade balance condition and dividing both sides by L_i we get our third key equation,

$$w_i s_{mi} + p_{mi} q_{mi} = p_{mi} q_i F_i \tag{15}$$

Plugging 13 into 15 we get

$$\left(\frac{\beta}{1-\beta}\right) p_{mi}q_{mi} + p_{mi}q_{mi} = p_{mi}q_iF_i$$

from which we get

$$q_{mi} = (1-\beta)q_iF_i \tag{16}$$

and using $q_{mi} + q_{fi} = q_i$ we get

$$q_{fi} = q_i(1-(1-\beta)F_i) \tag{17}$$

Plugging back into 15 we get $\frac{w_i s_{mi}}{p_{mi}} = \beta q_i F_i$. Using $s_{fi} + s_{mi} = 1$, this implies

$$w_i(1-s_{fi}) = \beta p_{mi}q_iF_i \tag{18}$$

From 12 and 17 we get

$$w_i s_{fi} = \left(\frac{\alpha}{1-\alpha}\right) p_{mi} q_i (1 - (1-\beta)F_i) \tag{19}$$

and from 18 and 19 we get $w_i s_{fi} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{w_i(1-s_{fi})}{\beta F_i}\right)(1-(1-\beta)F_i)$. Solving for s_{fi} we finally get

$$s_{fi} = \frac{\alpha(1-(1-\beta)F_i)}{(1-\alpha)\beta F_i + \alpha(1-(1-\beta)F_i)}$$

which is equation (3.16) in Alvarez-Lucas. This can also be written as

$$s_{fi} = \frac{\alpha - \alpha(1-\beta)F_i}{\alpha - (\alpha-\beta)F_i}$$

This is equal to α if $F_i = 1$, and also decreasing in F_i, so that if $F_i < 1$, then $s_{fi} > \alpha$: tariffs lead a country to use more labor directly in the production of final goods.

From (18) we get

$$p_{mi}q_i = \frac{w_i(1 - s_{fi})}{\beta F_i}$$

Plugging this into the trade balance condition and simplifying yields

$$L_i w_i(1 - s_{fi}(w)) = \sum_j L_j \frac{w_j(1 - s_{fi}(w))}{F_j(w)} D_{ji}(w)\omega_{ji}$$

Alvarez and Lucas then express this as an excess demand system,

$$Z_i(w) = \frac{1}{w_i}\left[\sum_j L_j \frac{w_j(1 - s_{fi}(w))}{F_j(w)} D_{ji}(w)\omega_{ji} - L_i w_i(1 - s_{fi}(w))\right]$$

The equilibrium w is the one where $Z(w) = 0$. Given a solution w, then $p_m(w)$, $D(w)$, $F(w)$, $s_f(w)$ and then obtain $q_f(w)$ and $q_m(w)$ from equations 12 and 13.