

Econ 580, Lecture Notes 11

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The paper is Lucas, "The Industrial Revolution: Past and Present" in **Lectures on Economic Growth**

The goal is a unified model where there is a Malthusian trap and also an equilibrium with endogenous growth.

Key ingredients are (1) preferences that embody a quantity vs. quality tradeoff affecting fertility,

$$W(c, n, u) = (1 - \beta) \ln(c) + \eta \ln(n) + \beta u.$$

Note that it's not like in BMT where we would have $\beta(n)nu$ instead of βu .

(2) a production function that results from the optimal use of two alternative technologies (a land using technology with $y = Ax^\alpha l^{1-\alpha}$ and a "growth" technology with $y = Bh l$),

$$F(x, h, l) = \max_{\theta \in [0, l]} [Ax^\alpha \theta^{1-\alpha} + Bh(l - \theta)].$$

(3) a cost of rearing children that is linear in *output*,

$$c + kn \leq F(x, h, l).$$

(4) a technology for human capital accumulation that uses r units of *labor* together with current human capital according to

$$h_{t+1} = h_t \varphi(r_t).$$

The time budget constraint is $l + rn = 1$. The law of motion for x_t is $x_{t+1} = x_t/n_t$.

The Bellman equation for the general problem is

$$v(x, h) = \max_{n, r} W(F(x, h, 1 - rn) - kn, n, v(x/n, h\varphi(r)))$$

Let's focus first on the environment which generates the Malthusian trap, where $r = 0$ and $\theta = l$. The Bellman equation is

$$v(x) = \max_{c, n} W(f(x) - kn, n, v(x/n))$$

From the F.O.C. and the envelope condition, using $u_t = v(x_t)$ and $q_t = v'(x_t)$, the full dynamic system is (I use the following shorthand: $W_{i,t} = W_i(c_t, n_t, u_{t+1})$ for $i = c, n, u$) $u_t = W(c_t, n_t, u_{t+1})$ together with

$$kW_{c,t} = W_{n,t} - W_{u,t}q_{t+1}x_{t+1} \left(\frac{1}{n_t^2} \right),$$

$$q_t = W_{c,t}f'(x_t) + W_{u,t}q_{t+1} \left(\frac{1}{n_t} \right),$$

$$x_{t+1} = x_t/n_t,$$

$$c_t + kn_t = f(x_t).$$

These are five equations in five variables, u_t , q_t , x_t , c_t and n_t .

A steady state has $n = 1$ and constant levels of u , q , x and c . Together with $u = W(c, n, u)$, q , x , and c are determined by

$$k \left(\frac{1 - \beta}{c} \right) = \eta - \beta qx,$$

$$q = \left(\frac{1 - \beta}{c} \right) A\alpha x^{\alpha-1} + \beta q,$$

and

$$c + k = Ax^\alpha.$$

The second equation implies

$$\begin{aligned}(1 - \beta)q &= \left(\frac{1 - \beta}{c}\right) A\alpha x^{\alpha-1} \\ &\rightarrow q = A\alpha x^{\alpha-1} c^{-1}.\end{aligned}$$

Plugging into the first equation yields

$$\begin{aligned}k \left(\frac{1 - \beta}{c}\right) &= \eta - \beta A\alpha x^{\alpha} c^{-1} \\ &\rightarrow k(1 - \beta) = \eta c - \beta A\alpha x^{\alpha} \\ &\rightarrow Ax^{\alpha} = \eta c / \alpha\beta - k(1 - \beta) / \alpha\beta\end{aligned}$$

Plugging into the third equation finally gives

$$\begin{aligned}c + k &= \eta c / \alpha\beta - k(1 - \beta) / \alpha\beta \\ &\rightarrow c \left(\frac{\eta}{\alpha\beta} - 1\right) = k \left(1 + \frac{1 - \beta}{\alpha\beta}\right) \\ &\rightarrow c_e = \frac{1 - \beta + \alpha\beta}{\eta - \alpha\beta} k\end{aligned}$$

The values of x can then be obtained from $c + k = Ax^{\alpha}$.

The important point to note here is that c_e doesn't depend on A . This is the way in which Lucas captures the

Malthusian trap: higher A just leads to a higher population and lower x in such a way that Ax^α remains constant.

Consumption depends on discounting β , the value of having children (higher η implies lower c_e), the cost of having children (higher k leads to higher c_e), and α .

Why is it that all A is "used" to increase population and not for higher consumption? Let's follow Lucas and start with the hunter-gatherer society where y is exogenous. Then the F.O.C. evaluated at steady state (where $n = 1$) is

$$kW_c(c, 1, u) = W_n(c, 1, u),$$

with $u = W(c, 1, u)$. The solution depends only on k and η and discounting. There is only one level of c consistent with choosing $n = 1$.

To see this more formally, let $u = g(c)$ be the solution of $u = W(c, 1, u)$. Assuming that $W_u \in (0, 1)$, then this is well defined and increasing function.

Letting

$$m(c) = \frac{W_n(c, 1, g(c))}{W_c(c, 1, g(c))},$$

then the equilibrium is determined by

$$m(c) = k$$

For the economy with private property we have

$$q_t = W_{c,t}f'(x_t) + W_{u,t}q_{t+1} \left(\frac{1}{n_t} \right)$$

so in steady state

$$q = \frac{W_c f'(x)}{1 - W_u}.$$

Plugging into the F.O.C. ($kW_c = W_n - W_u qx/n^2$)

$$k = m(c) - \frac{W_u}{1 - W_u} f'(x)x.$$

Lucas argues that $(1 - W_u)/W_u$ is like a discount rate and writes it as $\rho(c)$, hence we have

$$m(c) = k + \frac{f'(x)x}{\rho(c)}.$$

Note: to see that $(1 - W_u)/W_u$ is like a discount rate, note that if $u = W(c, n, u')$ then the utility of the present generation increases with the utility of the future generation according to $du = W_u du'$. But this implies that $du'/du = 1/W_u$. The discount rate is $du'/du - 1 = (1 - W_u)/W_u$.

If $f(x) = Ax^\alpha$ then land rents are $xf'(x) = \alpha f(x) = \alpha(c + k)$, and we have

$$m(c) = k + \frac{\alpha(c + k)}{\rho(c)}.$$

The solution implies that c is independent of A .

Note: if $f(x)$ is not CD then α would depend on x , so A would affect c through this channel.

Now consider the growth "steady state," where the land-using technology is "discarded," so $\theta = 0$, and n and r are constant. The household's Bellman equation is

$$v(h) = \max_{c,n,r} W(c, n, v(h'))$$

subject to

$$\begin{aligned} c + kn &= Bh l \\ h' &= h\varphi(r) \\ l + rn &= 1 \end{aligned}$$

or

$$v(h) = \max_{n,r} W(Bh - (Bhr + k)n, n, v(h\varphi(r)))$$

For large enough h , the cost kn of children becomes negligible, so we can write the Bellman equation as

$$v(h) = \max_{n,r} W(Bh(1 - rn), n, v(h\varphi(r)))$$

Assuming that $W(c, n, u)$ is homogenous of degree one in c and u (as it is with $W = c^{1-\beta} n^\eta u^\beta$) then $v(h) = Dh$ for D that satisfies

$$D = \max_{n,r} W(B(1 - rn), n, D\varphi(r))$$

The F.O.C. are

$$\begin{aligned}W_n &= W_c B r \\W_u D\varphi'(r) &= W_c B n\end{aligned}$$

With $W = c^{1-\beta} n^\eta u^\beta$ we have from $W_n = W_c B r$ that

$$\begin{aligned}\frac{\eta}{n} &= \frac{(1-\beta)Br}{B(1-rn)} \\&\rightarrow \eta(1-rn) = (1-\beta)rn \\&\rightarrow rn = \frac{\eta}{1-\beta+\eta}\end{aligned}$$

for the time expenditure in educating children, and $W_u D\varphi'(r) = W_c B n$ implies

$$\begin{aligned}\frac{\beta D\varphi'(r)}{D\varphi(r)} &= \frac{(1-\beta)Bn}{B(1-rn)} \\&\rightarrow \frac{\beta\varphi'(r)}{\varphi(r)} = \frac{(1-\beta)n}{1-rn}\end{aligned}$$

Letting $\varepsilon(r) \equiv \varphi'(r)r/\varphi(r)$ then

$$\beta(1-rn)\varepsilon(r) = (1-\beta)rn$$

Together with

$$rn = \frac{\eta}{1 - \beta + \eta}$$

we determine r and n . We have

$$rn = \frac{\beta\varepsilon(r)}{1 - \beta + \varepsilon(r)\beta}$$

hence we need $\varepsilon(r) = \eta/\beta$. This determines r implicitly (note that if ε is fixed, this doesn't have a solution... no steady state growth) and the constant endogenous growth rate $h_{t+1}/h_t = \varphi(r)$ for all t . This is like an AK model with endogenous population growth.

So now we have a Malthusian and a "modern growth" steady state. The question is about transition dynamics. This is not fully worked out, but the idea is that starting with some positive (exogenous) level of h , an increase in B would trigger the transition.