

Econ 580, Lecture Notes 12

Andres Rodriguez-Clare

Penn State University

April 20, 2009

The paper is Galor, "From Stagnation to Growth: Unified Growth Theory" in the Handbook of Economic Growth

Output is produced from human capital and land according to

$$Y_t = (h_t l_t L_t)^\alpha (A_t X_t)^{1-\alpha}$$

Using $x_t = A_t X_t / L_t$ then we can write

$$y_t = (h_t l_t)^\alpha x_t^{1-\alpha}$$

No property rights over land so

$$w_t = y_t / h_t l_t = (x_t / h_t l_t)^{1-\alpha}$$

Preferences over consumption and quantity and quality of children,

$$u^t = c_t^{1-\gamma} (n_t h_{t+1})^\gamma$$

with the constraint that $c_t \geq \tilde{c} > 0$.

Time budget constraint is

$$l_t + n_t(\tau + e_{t+1}) \leq 1$$

Multiplying by $w_t h_t$ we get

$$w_t h_t n_t(\tau + e_{t+1}) \leq (1 - l_t) w_t h_t$$

Since $c_t \leq w_t h_t l_t$ then using $z_t \equiv w_t h_t$ we can write

$$w_t h_t n_t(\tau + e_{t+1}) + c_t \leq z_t$$

or

$$c_t \leq w_t h_t (1 - n_t(\tau + e_{t+1}))$$

Letting

$$g_{t+1} = \frac{A_{t+1} - A_t}{A_t}$$

it is assumed that human capital depreciates faster with g_t ,

$$h_{t+1} = h(e_{t+1}, g_{t+1})$$

with

$$h_e > 0, h_{ee} < 0, h_g < 0, h_{gg} > 0, h_{eg} > 0, \\ h(0, 0) = 1, h_e(0, 0) = 0, \text{ and } \lim_{g \rightarrow \infty} h(0, g) = 0$$

The optimization problem for members of generation t is

$$\max_{n_t, e_{t+1}} (w_t h_t (1 - n_t (\tau + e_{t+1}))^{1-\gamma} (n_t h(e_{t+1}, g_{t+1}))^\gamma)$$

subject to

$$\begin{aligned} w_t h_t (1 - n_t (\tau + e_{t+1})) &\geq \tilde{c} \\ (n_t, e_{t+1}) &\geq 0 \end{aligned}$$

Playing with F.O.C. implies that if consumption constraint is not binding then

$$n_t (\tau + e_{t+1}) = \gamma$$

while otherwise $c = \tilde{c}$, which implies that

$$\begin{aligned} 1 - n_t (\tau + e_{t+1}) &= \tilde{c}/z_t \\ \implies n_t (\tau + e_{t+1}) &= 1 - \tilde{c}/z_t \end{aligned}$$

The consumption constraint is not binding as long as

$$z_t(\mathbf{1} - n_t(\tau + e_{t+1})) \geq \tilde{c}$$

or

$$z_t \geq \tilde{z} \equiv \frac{\tilde{c}}{\mathbf{1} - \gamma}$$

If $z < \tilde{z}$ then we have a corner solution and as z increases c stays fixed at \tilde{c} while $n_t(\tau + e_{t+1})$ increases towards γ . After $z \geq \tilde{z}$ then $n_t(\tau + e_{t+1})$ stays at γ and c grows proportionally with z ,

$$c_t = z_t(\mathbf{1} - \gamma)$$

What about the composition of n_t and e_{t+1} given $n_t(\tau + e_{t+1})$? The F.O.C.'s imply that

$$\frac{h_e(e_{t+1}, g_{t+1})}{h(e_{t+1}, g_{t+1})} = \frac{1}{\tau + e_{t+1}}$$

This yields a function $e_{t+1} = e(g_{t+1})$ with $e(g) = 0$ for $g \leq \hat{g}$ and otherwise $e' > 0$ and $e'' < 0$. Here we have

$$\frac{h_e(0, \hat{g})}{h(0, \hat{g})} = \frac{1}{\tau}$$

Note in particular that e_{t+1} does not depend directly on z_t .

We now have

$$n_t = \begin{cases} \frac{\gamma}{\tau + e(g_{t+1})} \equiv n^b(g_{t+1}) & \text{if } z_t \geq \tilde{z} \\ \frac{1 - [\tilde{c}/z_t]}{\tau + e(g_{t+1})} \equiv n^a(g_{t+1}, z(e_t, g_t, x_t)) & \text{if } z_t \leq \tilde{z} \end{cases}$$

where

$$\begin{aligned} z_t &\equiv w_t h_t = (x_t / h(e_t, g_t))^{1-\alpha} h(e_t, g_t) \\ &= x_t^{1-\alpha} h(e_t, g_t)^\alpha \equiv z(e_t, g_t, x_t) \end{aligned}$$

Some observations:

1. An increase in g_{t+1} reduces n_t and increases e_{t+1}
- higher growth leads to higher quality and lower quantity of children
2. If consumption constraint is binding then an increase in z_t increases n_t but has no effect on e_{t+1}
3. If consumption constraint is not binding then an increase in z_t has no effect on n_t or e_{t+1}

Finally, it is assumed that technological progress depends on human capital and population,

$$g_{t+1} = g(e_t, L_t)$$

with g increasing and strictly concave and $g(0, L_t) = 0$ for some positive L_t .

Let's focus first on the Malthusian steady state, which has $e_t = 0$, $c_t < \tilde{c}$, and $g_{t+1} = g(0, L_t)$. But what is happening with x_t and L_t ?

We know that $L_{t+1} = n_t L_t$ and

$$n_t = \frac{1 - (\tilde{c}/z_t)}{\tau}$$

so

$$x_{t+1} = \frac{[1 + g(0, L_t)] \tau}{1 - \tilde{c} / [x_t^{1-\alpha} h(0, g(0, L_t))^\alpha]} x_t$$

For L_t sufficiently low $g(0, L_t) = 0$ hence $h(0, g(0, L_t)) = 1$ so

$$x_{t+1} = \frac{\tau}{1 - \tilde{c} x_t^{\alpha-1}} x_t$$

This implies that steady state x is given implicitly by

$$\tau = 1 - \tilde{c} x^{\alpha-1}$$

All variables would be stagnant here and there would be no take-off to modern growth.

Is this stable? The derivative of $x_{t+1} = f(x_t)$ is

$$f'(x_t) = \frac{\tau}{1 - \tilde{c}x_t^{\alpha-1}} + (\alpha - 1) \frac{\tau x_t}{(1 - \tilde{c}x_t^{\alpha-1})^2} \tilde{c}x_t^{\alpha-2}$$

Evaluated at the steady state this is

$$\begin{aligned} f'(x) &= 1 - \frac{1 - \alpha}{\tau} \tilde{c}x^{\alpha-1} \\ &= 1 - \frac{1 - \alpha}{\tau} (1 - \tau) \end{aligned}$$

For stability we need

$$-1 < 1 - \frac{1 - \alpha}{\tau} (1 - \tau) < 1$$

The RHS is obvious, while the LHS entails

$$-2 < -\frac{1 - \alpha}{\tau} (1 - \tau)$$

or

$$2\tau > (1 - \alpha)(1 - \tau)$$

For this to be satisfied for all α need $\tau > 1/3$.

Assume instead that $g(0, 0) = 0$. Then the we will have a "conditional" (on L) steady state, but L_t will be growing over time.

Now disregard the $c_t \geq \tilde{c}$ constraint. We know that

$$L_{t+1} = n^b(g(e_t, L_t))L_t \equiv \frac{\gamma}{\tau + e(g(e_t, L_t))}L_t,$$

$$e_{t+1} = e(g(e_t, L_t)),$$

and

$$x_{t+1} = \frac{g(e_t, L_t)}{n^b(g(e_t, L_t))}x_t.$$

Note that an increase in e_t implies a higher g , which in turn implies a (weakly) higher e_{t+1} .

The assumptions made imply that for low L there is a unique level of e s.t. $e = e(g(e, L))$ given by $e = 0$, whereas for high L there is a unique but positive level of e s.t. $e = e(g(e, L))$. These two fixed points are stable.

For intermediate range of L there are three such fixed points. One such point has $e = 0$. The middle point is unstable in the sense that $e'(g(e, L))g_e(e, L) > 1$.

For high enough L but for e_t lower than the "good" steady state level of e the equilibrium entails an increasing e_t while population growth is falling because

$$n^b(g(e_t, L_t)) = \frac{\gamma}{\tau + e(g(e_t, L_t))}$$

It's not clear what happens in the limit.

As long as $g(0, 0) = 0$, the system has a built in mechanism to take us from the Malthusian "conditional" steady state to the era of modern growth, undergoing a demographic transition in the process.