Econ 580
Lecture Notes 14

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Infinite-horizon, overlapping generations model of continuum measure one of agents that live for two periods.

Discrete time indexed by t.

New technology every period, with $\gamma^t f(N, Z)$, $\gamma > 1$ (growth).

$f(N, Z)$ is CRS and strictly increasing and concave in N for each Z. This impil.es that $f(N, 0) = \omega_0 N$ where $\omega_0 \geq 0$.

Note: complementarities.

Notation: t and τ

Preferences: $u(c_1, c_2) = c_1 + \beta c_2$, with $\beta \in]0, 1[$.

Free lending and borrowing among young, hence the interest rate is pinned down by β, $\beta(1 + r) = 1$.

$\mu_t(\tau)$ is mass of old agents with experience in vintage τ - μ_t is the state of the economy

Evolution of the economy: letting $N_1(t,\tau)$ be number of young workers in τ at t, then

$$\mu_{t+1}(\tau+1) = N_1(t,\tau)$$

Let $N_2(t,\tau)$ be number of old workers that work as unskilled workers in τ at t.

Let $Z(t,\tau)$ denote skilled labor input in τ at t, then

$$Z(t\,,\tau) \leq \mu_t(\tau)$$

Let wages for unskilled and skilled workers by $w(t,\tau)$ and $v(t,\tau)$.

Old workers: if $Z(t,\tau) > 0$ then $v(t,\tau) \geq w(t,s)$ for all s. If $v(t,\tau) > w(t,s)$ then $Z(t,\tau) = \mu_t(\tau)$. If $N_2(t,\tau) > 0$ then $w(t,\tau) \geq w(t,s)$ for all s.

Young workers:

$$\begin{aligned} & w(t,\tau)+\beta\max\{v(t+1,\tau+1),w(t+1,l)\mid l\geq 0\} \\ \geq\ & w(t,s)+\beta\max\{v(t+1,s+1),w(t+1,l)\mid l\geq 0\} \end{aligned}$$

for all τ, s, and t s.t. $N_1(t,\tau)>0$.

Zero profits for profit-maximizing firms operating vintage τ at time t implies

$$\max_{N,Z}\left\{\gamma^{t-\tau}f(N,Z)-w(t,\tau)N-v(t,\tau)Z\right\}=0$$

for all $\tau\geq 1$ for all t

For $\tau=0$ we have $Z=0$ hence this implies

$$\max_{N}\left\{\gamma^{t}\omega_0 N-w(t,0)N\right\}=0$$

whicn implies that $w(t,0)=\gamma^t\omega_0$ - note that there is growth with $\gamma>1$.

A competitive equilibrium is a collection of wage functions $w(t,\tau)$ and $v(t,\tau)$; employment functions $N_1(t,\tau)$, $N_2(t,\tau)$ and $Z(t,\tau)$; and a sequence of distribution functions $\{\mu_t\}$ such that

1. Young workers are indifferent among vintages for which $N_1(t, \tau) > 0$.

2. Old workers maximize their income: $Z(t, \tau) > 0$ implies $v(t, \tau) \geq w(t, s)$ for all s; $v(t, \tau) > w(t, s)$ for all s implies $Z(t, \tau) = \mu_t(\tau)$; and $N_2(t, \tau) > 0$ then $w(t, \tau) \geq w(t, s)$ for all s.

3. Profit maximization: employment functions satisfy restrictions above.

4. Resource constraints: $\sum_\tau N_1(t, \tau) = 1$,
$$\sum_\tau [N_2(t, \tau) + Z(t\,, \tau)] = 1$$
for all t and $\mu_{t+1}(\tau+1) = N_1(t, \tau)$ and $Z(t\,, \tau) \leq \mu_t(\tau)$

Focus on stationary equilibrium, where all allocation variables $(\mu_t, N_1(t,\tau),\ N_2(t,\tau),\ Z(t,\tau))$ are stationary - don't depend on t - and all nominal variables $w(t,\tau)$ and $v(t,\tau)$ are constant except for growth at rate γ: $w(t,\tau) = \gamma^t w_\tau$ and $v(t,\tau) = \gamma^t v_\tau$.

Consider the allocation where only the new technology is used: $N_{1\tau} = N_{2\tau} = 0$ for $\tau \geq 1$. The wage would be ω_0. To check whether this an equilibrium, consider the deviation where the old work as skilled workers in vintage $\tau = 1$. A firm that does this would have to pay unskilled and skilled workers a wage of ω_0. Let

$$c(w,v) \equiv \min_{N,Z} \{wN + vZ \text{ s.t. } f(N,Z) = 1\}$$

Then ω_0 is defined by $c(\omega_0, \infty) = 1$ so we want to know whether $\gamma c(\omega_0, \omega_0) = \gamma\omega_0 c(1,1) \geq 1$. If $\gamma\omega_0 c(1,1) < 1$, then the postulated allocation is not an equilibrium.

CES. Imagine that

$$f(N,Z) = \left(N^{(\sigma-1)/\sigma} + Z^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}$$

with $\sigma > 1$. The unit cost associated with this function is

$$c(w, v) = \left(w^{1-\sigma} + v^{1-\sigma}\right)^{1/(1-\sigma)}$$

Note that $c(\omega_0, \infty) = \omega_0$, hence $c(\omega_0, \infty) = 1$ implies $\omega_0 = 1$, and $\gamma\omega_0 c(1, 1) \geq 1$ is equivalent to

$$2^{1/(1-\sigma)} \geq 1/\gamma$$

If γ is very high then this will be satisfied. But if $\sigma \to 1$ then $2^{1/(1-\sigma)} \to 0$ hence this cannot be satisfied. This implies that if N and Z are very complementary, then the postulated allocation cannot be an equilibrium.

Imagine that $2^{1/(1-\sigma)} = 1/\gamma$. Then both technologies can be operated, paying all workers a wage of ω_0, with $n_1 = 1$. Z_1 can be any number between 0 and N_{10}, with $N_{11} = Z_1$. This is a case of non-uniqueness (knife-edge).

Let's assume that $\gamma c(\omega_0, \omega_0) < 1$ so that $N_{1\tau} = N_{2\tau} = 0$ for $\tau \geq 1$ is not an equilibrium. Imagine also that only vintages $\tau = 0$ and $\tau = 1$ can be used. What is the equilibrium allocation?

We must have $Z_1 = N_0$ (otherwise $v_1 = \omega_0$ and there would be positive profits in operating vintage 1). We also need

$$\omega_0 + \beta\gamma v_1 = w_1 + \beta\gamma w_1$$

and

$$\gamma c(w_1, v_1) = 1$$

as well as $c(\omega_0, \infty) = 1$. (In the terminology of Chari and Hopenhayn, this is like having $T = 2$ and $Z_2 = 0$.)

In equilibrium we must have that $v_1 > w_1$. Proof by contradiction: (1) if $v_1 = w_1$ then $\omega_0 = v_1 = w_1$ and hence $\gamma c(w_1, v_1) = \gamma c(\omega_0, \omega_0) < 1$. (2) If $v_1 < w_1$ then $\omega_0 > w_1$, hence $v_1 < w_1 < \omega_0$, hence $\gamma c(w_1, v_1) < \gamma c(\omega_0, \omega_0) < 1$.

Hence necessarily $v_1 > w_1 > \omega_0$. We need to have $Z_1 = N_{10}$: old workers that are skilled in vintage 1 work as skilled workers in that vintage.

We can transform the previous system into a system of two equations in w_1 and v_1/w_1,

$$\omega_0/w_1 + \beta\gamma v_1/w_1 = 1 + \beta\gamma$$

$$\gamma w_1 c(1, v_1/w_1) = 1$$

In the CES case this second equation is

$$\gamma^{1-\sigma} w_1^{1-\sigma} \left(1 + \left(\frac{v_1}{w_1}\right)^{1-\sigma}\right) = 1$$

or

$$\gamma 2^{1/(1-\sigma)} w_1 \left(\frac{1}{2} + \frac{1}{2}\left(\frac{v_1}{w_1}\right)^{1-\sigma}\right)^{1/(1-\sigma)} = 1$$

Using

$$\omega_0/w_1 = 1 + \beta\gamma - \beta\gamma v_1/w_1$$

and $\omega_0 = 1$ then

$$\frac{\gamma 2^{1/(1-\sigma)}}{1 + \beta\gamma - \beta\gamma v_1/w_1} \left(\frac{1}{2} + \frac{1}{2}\left(\frac{v_1}{w_1}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = 1$$

Since the LHS is increasing in v_1/w_1 then as $\gamma 2^{1/(1-\sigma)}$ increases then v_1/w_1 decreases. Since for $\gamma 2^{1/(1-\sigma)} = 1$ we have $v_1 = w_1 = \omega_0$ then this shows that as $\gamma 2^{1/(1-\sigma)}$ falls below 1 then v_1/w_1 increases above one, so $v_1 > w_1$. This implies that $n_1 < 1$, so $N_{11} < N_{10}$. Moreover, as $\gamma 2^{1/(1-\sigma)}$ continues to fall, n_1 increases and so does N_{11}/N_{10}, and diffusion becomes slower.

Now allow the use of vintage 2. Is the previous allocation an equilibrium? To operate vintage 2 firms need to pay skilled workers a wage of $v_2 = w_1$ and then pay the unskilled a wage w_2 satisfying

$$w_2 + \beta\gamma w_1 = w_1 + \beta\gamma w_1$$

hence $w_2 = w_1$. So the previous allocation is an equilibrium if

$$\gamma^2 c(w_1, w_1) = \gamma^2 w_1 c(1,1) \geq 1$$

Since $w_1 > \omega_0$ and $\gamma > 1$ then this condition is easier to satisfy than $\gamma\omega_0 c(1,1) > 1$. So we could have $\gamma\omega_0 c(1,1) < 1$ and yet $\gamma^2 w_1 c(1,1) \geq 1$.

If $\gamma w_1 c(1,1) < 1$ then vintage 2 will also be operated in equilibrium. Again, imagine that vintage 3 cannot be operated and solve for the equilibrium. Now we have

$$\omega_0 + \beta\gamma v_1 = w_1 + \beta\gamma v_2 = w_2 + \beta\gamma w_2$$

and

$$c(\omega_0, \infty) = \gamma c(w_1, v_1) = \gamma^2 c(w_2, v_2) = 1$$

together with the restriction that $v_2 \geq w_2$. One can show that the solution to this system of 4 equations in v_1, v_2, w_1 and w_2 satisfies

$$v_1 > v_2 \geq w_2 \geq w_1 > \omega_0$$

Proof: First note that the condition

$$w_1 + \beta\gamma v_2 = w_2 + \beta\gamma w_2$$

immediately implies that $v_2 \geq w_2 \geq w_1$ because if $v_2 \geq w_2$ then necessarily $w_1 \leq w_2$. But then the condition

$$\gamma c(w_1, v_1) = \gamma^2 c(w_2, v_2)$$

together with $\gamma > 1$ and the fact that $c(w, v)$ is increasing in both arguments implies that $v_1 > v_2$. The condition

$$w_0 + \beta\gamma v_1 = w_1 + \beta\gamma v_2$$

then immediately implies that $w_1 > w_0$. **Q.E.D.**

This proof can be extended to T vintages with

$$\omega_0 + \beta\gamma v_1 = w_1 + \beta\gamma v_2 = ... = w_{T-1} + \beta\gamma w_{T-1}$$

(note that old workers skilled in $\tau = T - 1$ work as unskilled in vintage $T - 1$) and

$$\begin{aligned} c(\omega_0, \infty) &= \gamma c(w_1, v_1) = \\ ... &= \gamma^{T-1} c(w_{T-1}, v_{T-1}) = 1 < \gamma^T c(w_T, w_T) \end{aligned}$$

with $v_{T-1} \geq w_{T-1}$.

Note: there is also a case with $\gamma^T (w_T, w_T) = 1$, but this can be ignored as it is a knife-edge case.

Chari and Hopenhayn establish existence and (almost always) uniqueness of wages that satisfy these conditions. The equilibrium allocation is then given by

$$\frac{c_w(w_\tau, v_\tau)}{c_v(w_\tau, v_\tau)} = n_\tau = \frac{N_{1\tau}}{N_{2\tau}} = \frac{N_{1\tau}}{N_{1\tau-1}}$$

for $\tau \leq T - 1$. For $\tau = T - 1$ we have

$$\frac{c_w(w_{T-1}, v_{T-1})}{c_v(w_{T-1}, v_{T-1})} = n_{T-1} = 2\frac{N_{1T-1}}{N_{1T-2}}$$

So given N_{10} and wages we construct the sequence N_{10}, N_{11}, N_{12}, ..., N_{1T-1}. N_{10} has to be such that the sum of this sequence adds up to 1.

Note for future reference that $\mu_\tau = n_{\tau-1}\mu_{\tau-1}$. This follows from

$$\begin{aligned} N_{2\tau} &= N_{1\tau-1} \\ &\implies N_{2\tau} = \left(\frac{N_{1\tau-1}}{N_{2\tau-1}}\right) N_{2\tau-1} = n_{\tau-1} N_{2\tau-1} \end{aligned}$$

Some implications:

n_τ is decreasing in τ: older technologies are operated with a higher ratio of skilled to unskilled workers.

There is a vintage R s.t. for all $\tau \leq R$, $\mu_\tau \geq \mu_{\tau-1}$ and for $\tau > R$, $\mu_\tau \leq \mu_{\tau-1}$. Note: R could be zero (!?).

The distribution associated with a higher growth rate $\gamma' > \gamma$ is dominated in the sense of stochastic dominance by the original distribution, i.e. the distribution of skilled workers shifts to more recent vintages.

The earnings profile becomes flatter as the growth rate of the economy increases - higher growth implies a lower experience premium.