Econ 580, Lecture Notes on Becker, Murphy and Tamura

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To understand why $A(T - vn_{t+1})$ is the rate of return to h, imagine that parents at time t increase h by one unit. Then there would be an increase in H_{t+1} of $A(H^0 + H_t)$ which tomorrow can be used to increase consumption c_{t+1} by $DA(T - vn_{t+1})(H^0 + H_t)$ (note: one could think that instead of $T - vn_{t+1}$ we should have l_{t+1} here... but think about the Envelope Theorem) while the loss would be that now l_t would go down by n_t, which has a consumption cost today of $Dn_t(H^0 + H)$. Thus, since tomorrow we have n_t people benefiting from the higher c_{t+1}, then the total increase in consumption tomorrow per unit of consumption sacrificed today is

$$n_t \frac{DA(T - vn_{t+1})(H^0 + H_t)}{Dn_t(H^0 + H_t)} = A(T - vn_{t+1})$$

The Euler equation (equation 9) then implies that

$$\frac{1}{a(n_t)} \left(\frac{c_{t+1}}{c_t} \right)^{1-\sigma} \geq R_{ht} = A(T - vn_{t+1})$$

with equality if investments in human capital are positive.

Let's understand how fertility decisions work in BMT and the presence of income and substitution effects. To do this, let's shut down human capital accumulation for now. Then we have no state variable, and the Bellman equation is simply

$$V = \max_n \left[(DH^0(T - vn) - fn)^\sigma / \sigma + \alpha n^{1-\varepsilon} V \right]$$

so parents choose n to maximize dynastic utility which is now simply

$$G(n) = \frac{(DH^0(T - vn) - fn)^\sigma / \sigma}{1 - \alpha n^{1-\varepsilon}}$$

Taking logs and differentiating w.r.t. n we get the first order condition

$$\frac{\sigma(DH^0 v + f)}{DH^0(T - vn_u) - fn_u} = \frac{\alpha(1 - \varepsilon) n_u^{-\varepsilon}}{1 - \alpha n_u^{1-\varepsilon}}$$

One can think of the LHS and the RHS as the marginal cost and marginal benefit of an increase in n, respectively. This is the same as equation (13) except for the D, but I insist they have a typo here. One can do comparative statics (assuming S.O.C.) holds and show that an increase in f or v leads to a drop in n. It is clear that if f and v are such that n_u is high then equation (12) will hold so that $n = n_u$ with $H = 0$ is a steady state equilibrium path.

But what is the effect of H^0 on n_u? There are two opposite effects (numerator vs denominator). The increase in the denominator corresponds to the fact that an increase in H^0 increases consumption given n - this decreases the marginal benefit of consumption because of diminishing marginal benefits of consumption. This is an income effect leading to an increase in n. Note that if $v = 0$ then this is the only effect and necessarily an increase in H^0 leads to an increase in n. But with $v > 0$ there is also a time cost of having children, and this time cost is more costly as H^0 increases - this is the substitution effect reflected in the fact that the numerator of the LHS increases with H^0.

Which of these effects dominates? We already know that if $v = 0$ then there is no substitution effect, so this implies that for the substitution effect to dominate we need v to be large. We can also see that if $f = 0$ then the two effects exactly cancel out, so that H^0 has no effect on n_u. Can we say something about how H^0 affects the LHS? Take logs of the LHS and then differentiate w.r.t. H^0 to get

$$\frac{Dv}{DH^0 v + f} - \frac{D(T - vn_u)}{DH^0(T - vn_u) - fn_u}$$

This has the same sign as

$$\begin{aligned} & v(DH^0(T - vn_u) - fn_u) - \left(DH^0 v + f\right)(T - vn_u) \\ = \; & vDH^0 T - v^2 DH^0 n_u - vfn_u - DH^0 vT + DH^0 v^2 n_u - fT + vfn_u \\ = \; & -fT \end{aligned}$$

so it is always negative, hence a higher H^0 leads to a lower marginal cost of children and hence an increase in n_u. The income effect always dominates.

Now let's see how human capital accumulation works by shutting down fertility decisions by setting $n_t = n$ for all t. Consumption is now

$$\begin{aligned} c_t &= Dl_t(H^0 + H_t) - fn \\ &= D\left(T - (v + h_t)n\right)(H^0 + H_t) - fn \\ &= D\left(T - (vn + H_{t+1}A^{-1}(H^0 + H_t)^{-1})n\right)(H^0 + H_t) - fn \\ &= D\left((T - vn)(H^0 + H_t) - H_{t+1}nA^{-1}\right) - fn \end{aligned}$$

so the Bellman equation can be rewritten as

$$V_t(H_t) = \max_{H_{t+1}} \left[\frac{\left(D\left((T - vn)(H^0 + H_t) - H_{t+1}nA^{-1}\right) - fn\right)^{\sigma}}{\sigma} + \alpha n^{1-\varepsilon} V_{t+1}(H_{t+1}) \right]$$

Differentiation w.r.t. H_{t+1} yields

$$-c_t^{\sigma-1}Dn/A + \alpha n^{1-\varepsilon}V'_{t+1} \leq 0$$

while the envelope theorem implies

$$V'_{t+1} = c_{t+1}^{\sigma-1}D(T-vn)$$

hence we have

$$-c_t^{\sigma-1}Dn/A + \alpha n^{1-\varepsilon}c_{t+1}^{\sigma-1}D(T-vn) \leq 0$$

or

$$-c_t^{\sigma-1}n/A + \alpha n^{1-\varepsilon}c_{t+1}^{\sigma-1}(T-vn) \leq 0$$

which is equivalent to equation (9) in BMT.

We have a corner solution with $h = 0$ if

$$A(T-vn) < \alpha^{-1}n^{\varepsilon}$$

A low A, a low T, a high v, a high n and a low α all help to get this result.

If $h > 0$ then we would have $c_t = c_{t+1}$ only in the knife edge case with $A(T-vn) = \alpha^{-1}n^{\varepsilon}$. Assuming this away, then in an interior equilibrium we must have growth given by

$$\frac{c_{t+1}}{c_t} = \left(\frac{A(T-vn)}{\alpha^{-1}n^{\varepsilon}}\right)^{1/(1-\sigma)}$$

Summarizing, if $A(T-vn) < \alpha^{-1}n^{\varepsilon}$ then we have $h_t = 0$ for all t, whereas if $A(T-vn) > \alpha^{-1}n^{\varepsilon}$ then we have growth as in the equation above. (Note that one can back out what h must be from this equation).

Now let's consider again the model with no human capital accumulation as above but assume that we start with some H_t that disappears next period (because $h = 0$ implies $H_{t+1} = 0$). The Bellman equation is simply

$$V_t = \max_n \left[(D(H^0+H)(T-vn) - fn)^{\sigma}/\sigma + \alpha n^{1-\varepsilon}V_{t+1}\right]$$

with

$$V_{t+1} = \max_n G(n) = G(n_u) \equiv \frac{c_u^{\sigma}}{\sigma(1-\alpha n^{1-\varepsilon})}$$

so parents at time t choose n to maximize

$$(D(H^0+H)(T-vn) - fn)^{\sigma}/\sigma + \alpha n^{1-\varepsilon}G(n_u)$$

The first order condition is

$$c^{\sigma-1}(D(H^0+H)v + f) = \alpha(1-\varepsilon)n^{-\varepsilon}\frac{c_u^{\sigma}}{\sigma(1-\alpha n_u^{1-\varepsilon})}$$

or

$$D(H^0+H)v+f=\alpha(1-\varepsilon)n^{-\varepsilon}\left(\frac{c_u}{c}\right)^{\sigma-1}\frac{c_u}{\sigma(1-\alpha n_u^{1-\varepsilon})}$$

A higher H implies a higher marginal cost of fertility, but also implies that c_u/c is lower and hence there is less "discounting" so the marginal benefit of fertility is higher.

We can rewrite this FOC as

$$\left(D(H^0+H)v+f\right)\left(D(H^0+H)(T-vn)-fn\right)^{\sigma-1}n^{\varepsilon}=\frac{\alpha(1-\varepsilon)c_u^{\sigma}}{\sigma(1-\alpha n_u^{1-\varepsilon})}$$

Differentiating the log of the RHS w.r.t. H yields

$$\frac{Dv}{D(H^0+H)v+f}+(\sigma-1)\frac{D(T-vn)}{D(H^0+H)(T-vn)-fn}$$

and the sign of this is equal to the sign of

$$\begin{aligned}
& D(H^0+H)(T-vn)v-fnv+(\sigma-1)(T-vn)\left(D(H^0+H)v+f\right)\\
= \ & \sigma D(H^0+H)(T-vn)v-fnv+(\sigma-1)(T-vn)f\\
= \ & \sigma D(H^0+H)(T-vn)v-fnv+\sigma(T-vn)f-Tf+vnf\\
= \ & \sigma D(H^0+H)(T-vn)v+\sigma(T-vn)f-Tf\\
= \ & \sigma(T-vn)\left[D(H^0+H)v-f\right]-Tf
\end{aligned}$$

If this expression is positive then n_t decreases as H_t increases. (Note: I was hoping to connect with equation 14, but this didn't happen!)

We can now use the different ideas reviewed so far to understand BMT's explanation of the model in page S23. First of all, if f and v are sufficiently high then n_u will be high so that equation (12) will hold and $H_t=0$ is a steady state. Second, if we happened to start with H_t just slightly positive, then the equilibrium path would have $n_t=\widetilde{n}(H_t)$ given implicitly by

$$\left(D(H^0+H_t)v+f\right)\left(D(H^0+H_t)(T-vn_t)-fn_t\right)^{\sigma-1}n_t^{\varepsilon}=\frac{\alpha(1-\varepsilon)c_u^{\sigma}}{\sigma(1-\alpha n_u^{1-\varepsilon})}$$

and $n_{t+1}=n_u$. The following inequality would ensure that $h_t=0$ (and $H_{t+1}=0$):

$$-(c_t(H_t,\widetilde{n}(H_t)))^{\sigma-1}\widetilde{n}(H_t)/A+\alpha\widetilde{n}(H_t)^{1-\varepsilon}c_u^{\sigma-1}(T-vn_u)<0$$

Thus, the steady state with no human capital accumulation is locally stable. Third, since the function $\widetilde{n}(H_t)$ becomes decreasing at some point, there is a level of H_t sufficiently high that we have

$$-(c_t(H_t,\widetilde{n}(H_t)))^{\sigma-1}\widetilde{n}(H_t)/A+\alpha\widetilde{n}(H_t)^{1-\varepsilon}c_u^{\sigma-1}(T-vn_u)>0$$

But when this inequality is just satisfied, the amount invested "is insufficient to maintain the stock of human capital, and the economy returns over time to the steady state" with no human capital accumulation. If the RHS of equation (18) is higher than one, there is a steady state with growth, and the economy converges to that steady state for a high enough level of H_t.