Econ 559
The AK Growth Model
Based on Acemoglu, Chapter 11

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Technology: $Y = AK$, so $y = Ak$, $y = Y/L$ and $k = K/L$.

Note that $f'(k) = A > 0$ as $k \to \infty$. This is the key to long run growth in this model. In fact, need more...

Assume CRRA preferences to allow for an equilibrium with constant growth rate of consumption,

$$\frac{\dot{c}(t)}{c(t)} = \frac{r - \rho}{\theta}$$

Since $r = R - \delta = A - \delta$, for growth need that $A - \delta > \rho$. If this condition is satisfied, then the growth rate (of c, k, y) is

$$g = \frac{A - \delta - \rho}{\theta}$$

For utility to be well defined and for the TC to be satisfied, need

$$\rho > (1-\theta)(A-\delta) + \theta n$$

This equivalent to imposing $\rho > (1-\theta)g + n$, the condition needed in the NGM.

No transition dynamics... why?

Wage is zero... but can also get this with

$$Y = F(K, L) = A(K^\rho + L^\rho)^{1/\rho}$$

as long as $\rho \in]0,1]$. In this case we have $f(k) = A(k^\rho + 1)^{1/\rho}$ and can easily show that $\lim_{k\to\infty} f'(k) = A$.

Now wages are positive, but the labor share converges to zero.

The AK Model with K and H

Now consider $Y = F(K, H)$ with no population growth and

$$\begin{aligned} \dot{a} &= ra + wh - c - i_h \\ \dot{h} &= i_h - \delta_h h \end{aligned}$$

Using $k = K/H$ we have

$$R = f'(k) \text{ and } w = f(k) - kf'(k)$$

In equilibrium need

$$w - \delta_h = r = R - \delta_k$$

but this requires

$$f(k) - kf'(k) - \delta_h = f'(k) - \delta_k$$

which defines a level k^* that has to hold at all times.

Assuming that

$$f'(k^*) > \rho + \delta_k$$

and

$$\rho > (1-\theta)(f'(k^*) - \delta)$$

The growth rate is

$$g = \frac{f'(k^*) - \rho - \delta_k}{\theta}$$

Again, no transition dynamics. Why?

Growth with Externalities

Now have $Y = F(K, AL)$ with $A = BK$ taken as given by firms.

This implies that $Y = F(1, BL)K \equiv \widetilde{A}K$, so this is like an AK model... except that externalities reduce the private return to capital below the social return, hence equilibrium growth is lower than optimal growth.

Specifically, we have

$$\begin{aligned} r &= F_K(K, AL) - \delta = F_K(1, BL) - \delta \\ w &= F(1, BL)k - F_K(1, BL)k \end{aligned}$$

and

$$g = \frac{1}{\theta}\left(F_K(1, BL) - \delta - \rho\right)$$

The optimal growth rate takes into account that the social marginal return of capital is

$$\begin{aligned} & F_K(1, BL) + BLF_L(1, BL) \\ = \; & F(1, BL) = \frac{dF(K, BKL)}{dK} \end{aligned}$$

hence

$$g^s = \frac{1}{\theta}(F(1, BL) - \delta - \rho)$$

Another possibility: $A = BK_G$, with $\dot{K}_G = \tau_K rK + \tau_L wL - \delta$.

Problem set #2, part A: derive growth given τ_K and τ_L and then derive the optimal tax policy in this economy.

The Two-Sector AK Model

Now consider

$$\begin{aligned} C &= BK_C^{\alpha} L_C^{1-\alpha} \\ \dot{K} &= I - \delta K \\ I &= AK_I \end{aligned}$$

Imagine that $K_I(t) = \kappa K(t)$ for all t and some $\kappa \in]0, 1[$. Then

$$g_K = \frac{\dot{K}_C}{K_C} = \frac{\dot{K}_I}{K_I} = \frac{\dot{K}}{K} = I/K - \delta = A\kappa - \delta$$

and

$$\frac{\dot{C}}{C} = \alpha \frac{\dot{K}_C}{K_C} = \alpha g_K$$

The savings rate is

$$\frac{p_I(t)I(t)}{p_C(t)C(t) + p_I(t)I(t)}$$

Letting $p_C(t) = 1$ then this is $\frac{p_I(t)I(t)/C(t)}{1+p_I(t)I(t)/C(t)}$. But note that

$$\begin{aligned} \frac{p_I(t)I(t)}{C(t)} &= \frac{p_I(t)A\kappa K(t)}{B(1-\kappa)^\alpha K(t)^\alpha L^{1-\alpha}} \\ &= \frac{p_I(t)A\kappa K(t)^{1-\alpha}}{B(1-\kappa)^\alpha L^{1-\alpha}} \end{aligned}$$

What is $p_I(t)$? This is defined from the condition that the value of the marginal product of capital be equalized across the two sectors,

$$p_I A = \alpha B \left(\frac{L}{(1-\kappa)K} \right)^{1-\alpha}$$

Plugging above, then

$$\frac{p_I(t)I(t)}{C(t)} = \frac{p_I(t)A\kappa K(t)^{1-\alpha}}{B(1-\kappa)^\alpha L^{1-\alpha}} = \frac{\alpha\kappa}{(1-\kappa)}$$

which is constant. So a BGP is feasible.

Also note that labor share is

$$\frac{(1-\alpha)C(t)}{C(t)+p_I(t)I(t)} = \frac{1-\alpha}{1+p_I(t)I(t)/C(t)}$$

which is constant.

To get κ in equilibrium, need to use $g_C = \frac{1}{\theta}(r_C - \rho)$, where r_C is the interest rate in terms of consumption goods. But

$$r_C = A - \delta + \frac{\dot{p}_I}{p_I} = A - \delta - (1-\alpha)g_K$$

hence we have (assuming that $A - \delta - \rho > 0$)

$$\begin{aligned} g_C &= \frac{1}{\theta}(A - \delta - (1-\alpha)g_K - \rho) \\ \alpha\theta g_K &= A - \delta - (1-\alpha)g_K - \rho \\ g_K &= \frac{A-\delta-\rho}{1-\alpha(1-\theta)} \\ g_C &= \alpha\frac{A-\delta-\rho}{1-\alpha(1-\theta)} \end{aligned}$$

To get κ use $A\kappa - \delta = g_K$.

Note: for utility to be bounded, need to assume that

$$
\begin{aligned}
\rho &> \alpha(1-\theta)\frac{A-\delta-\rho}{1-\alpha(1-\theta)} \\
&\rightarrow \rho > \alpha(1-\theta)(A-\delta)
\end{aligned}
$$

A General Neoclassical Model of Sustained Growth

Consider a two sector model with physical and human capital defined by

$$
\begin{aligned}
C + I &= K_Y^{\alpha} H_Y^{1-\alpha} \\
\dot{K} &= BI - \delta_K K \\
\dot{H} &= K_H^{\gamma} H_H^{1-\gamma} - \delta_H H \\
K_Y + K_H &= K \\
H_Y + H_H &= H
\end{aligned}
$$

If $\alpha = 1$ then this is the AK model.

If $\alpha = \gamma$ then this is the AK model with K and H.

If $B = \delta_K = 0$ and $K(0) = 1$ then this corresponds to the two-sector growth model above but with human capital as the engine of growth.

Assume $B = 1$ and $\delta_K = \delta_H = \delta$. Problem set #2, part B: solve this model with CRRA preferences. Is there a steady state growth rate for any value of γ? For what value of γ is the growth rate higher?