Econ 559
Expanding Variety Models of Endogenous Growth
Based on Acemoglu, Chapter 13

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The Power of Love

Consider a model with

$$\begin{aligned} Y(t) &= (1-\beta)\widetilde{X}(t)^{1-\beta}L^{\beta}, \\ \widetilde{X}(t) &= \left(\int_0^{N(t)} X(v,t)^{\frac{\varepsilon-1}{\varepsilon}} dv\right)^{\frac{\varepsilon}{\varepsilon-1}}. \end{aligned}$$

We assume that $X(v,t)$ is produced at cost ψ of output. We can think of using output to produce $X(t)$ one to one and then using $X(t)$ to produce $X(v,t)$ at cost ψ. If $N(t)$ was exogenous, then we would just write $Y(t) = C(t) + X(t)$.

Since $X(v,t)$ enters symmetrically into the production of $\widetilde{X}(t)$ then it is optimal to set $X(v,t) = (1/\psi)X(t)/N(t)$, so that

$$\widetilde{X}(t) = (1/\psi)N(t)^{\frac{\varepsilon}{\varepsilon-1}}X(t)/N(t) = (1/\psi)N(t)^{\frac{1}{\varepsilon-1}}X(t)$$

and

$$Y(t) = (1/\psi)N(t)^{\frac{1-\beta}{\varepsilon-1}}X(t)^{1-\beta}L^{\beta}$$

Note Love of Variety with $\varepsilon > 1$.

Producers of final goods will spend $1-\beta$ of output buying inputs. Assume that there is a mark-up m over cost, so that $P(v,t) = m\psi$. Then we know that

$$(1-\beta)\,Y(t) = \widetilde{X}(t)P(t)$$

with

$$\begin{aligned} P(t)^{1-\varepsilon} &= \int_0^{N(t)} P(v,t)^{1-\varepsilon} dv \\ P(t) &= N(t)^{\frac{1}{1-\varepsilon}} m\psi \end{aligned}$$

so

$$\begin{aligned} (1-\beta)\,Y(t) &= (1/\psi)N(t)^{\frac{1}{\varepsilon-1}} X(t) N(t)^{\frac{1}{1-\varepsilon}} m\psi \\ &\implies X(t) = \left(\frac{1-\beta}{m}\right) Y(t) \end{aligned}$$

and hence

$$\begin{aligned} Y(t) &= N(t)^{\frac{1-\beta}{\varepsilon-1}} \left(\left(\frac{1-\beta}{m}\right) Y(t) \right)^{1-\beta} L^{\beta} \\ &\implies Y(t) = \alpha N(t)^{\gamma} L \end{aligned}$$

where $\alpha \equiv \left(\frac{1-\beta}{m}\right)^{(1-\beta)/\beta}$ and $\gamma \equiv \frac{1-\beta}{\varepsilon-1}\frac{1}{\beta}$.

If there is an exogenous process with $\dot{N}(t)/N(t) = g_N > 0$ then we have growth as in the NGM with exogenous technological change. But can we endogeneize g_N?

Consider two alternative models.

(a) Lab-Equipment Model

Here we assume that research is produced just like output, so that

$$C(t) + X(t) + Z(t) \leq Y(t)$$

Imagine that $\dot{N}(t) = \eta Z(t)$ and $Z(t) = zY(t)$. Then

$$\dot{N}(t)/N(t) = \eta z \alpha N(t)^{\gamma - 1} L$$

For a BGP need $\gamma = 1$, or

$$\begin{aligned} \frac{1-\beta}{\varepsilon - 1}\frac{1}{\beta} &= 1 \\ \implies \varepsilon &= 1/\beta \end{aligned}$$

This is a "knife-edge" condition needed for there to be BGP in the lab-equipment model.

The growth rate would be

$$g = \eta z \alpha L$$

Note "strong scale effect" - higher L leads to higher growth rate. Also higher investment in R&D (higher z) leads to higher growth, whereas a higher mark-up leads to lower growth.

(b) Cuasi-endogenous growth in the lab-equipment model

Imagine that $\varepsilon > 1/\beta$ and that there is growth in L at rate g_L. For a constant growth rate in $N(t)$ need

$$(\gamma - 1)\, g_N + g_L = 0$$

hence in steady state we have

$$g_N = \frac{g_L}{1-\gamma} > 0$$

Using $y(t) = Y(t)/L(t)$ in steady state we have

$$g = \dot{y}(t)/y(t) = \frac{\gamma g_L}{1-\gamma}$$

(c) R&D using only labor

Now we have $\dot{N} = \eta L_R$, with $L_R = zL$ and $Y(t) = \alpha N(t)^\gamma L_Y = \alpha(1-z)N(t)^\gamma L$. Then

$$\dot{N}(t)/N(t) = \eta z L/N(t)$$

This cannot remain positive and constant. Add spillovers: $\dot{N}(t) = \eta z L N(t)^\theta$.

Two possibilities:

(i) If $\theta = 1$ and $g_L = 0$ then

$$g_N = \eta z L \text{ and } g = \gamma \eta z L$$

Note: strong scale effects.

(ii) Cuasi-endogenous growth - assume $0 < \theta < 1$ and $g_L > 0$. Now have

$$g_N = \frac{g_L}{1-\theta} > 0 \text{ and } g = \frac{\gamma g_L}{1-\theta}$$

Lab-Equipment Model (Equilibrium Solution)

Continuous time with constant population that admits an infinitely lived representative agent with standard preferences with discount rate ρ and CRRA preferences with IES equal to θ.

Production function is

$$Y(t) = \frac{1}{1-\beta}\widetilde{X}(t)^{1-\beta}L^{\beta}$$

where

$$\widetilde{X}(t) = \left(\int_0^{N(t)} x(v,t)^{\frac{\varepsilon-1}{\varepsilon}}dv\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Here $\varepsilon > 1$ is the elasticity of substitution between varieties and $N(t)$ is the number (measure) of varieties available at time t. If we assume that $\varepsilon = 1/\beta$ then this simplifies to

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta}\right)L^{\beta}$$

Note CRS given $N(t)$.

Lab equipment because

$$C(t) + X(t) + Z(t) \leq Y(t)$$

with

$$\dot{N}(t) = \eta Z(t)$$

and $N(0)$. There is no aggregate uncertainty, but there could be idiosyncratic uncertainty.

Households hold a diversified portfolio of research labs, which - if successful - receive perpetual patents, behave as monopolists and charge prices (or rental price for capital) $p^x(v, t)$.

Final good producers choose $x(v,t)$ for $v \in [0, N(t)]$ to maximize

$$\frac{1}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta} dv\right) L^{\beta} - \int_0^{N(t)} p^x(v,t)x(v,t)dv - w(t)L$$

which implies (FOC) a demand schedule

$$x(v,t) = p^x(v,t)^{-\varepsilon} L = p^x(v,t)^{-1/\beta} L$$

Instantaneous profits for monopolists are

$$\pi(v,t) = p^x(v,t)x(v,t) - \psi x(v,t)$$

and the value of a blueprint for v is

$$V(v,t) = \int_t^{\infty} e^{-r(t,s)(s-t)} \pi(v,s) ds$$

where

$$r(t,s) = \frac{1}{s-t}\int_t^s r(s')ds'$$

Monopolists choose

$$p^x(v,t) = \frac{MC}{1 - 1/\varepsilon} = \frac{\psi}{1 - \beta}$$

Assuming that $\psi = 1 - \beta$ then $p^x(v,t) = 1$ for all v and t, and hence

$$x(v,t) = p^x(v,t)^{-1/\beta} L = L \text{ for all } v \text{ and } t$$

which implies that

$$\begin{aligned} \pi(v,t) &= p^x(v,t)x(v,t) - \psi x(v,t) \\ &= (1 - \psi) L = \beta L \text{ for all } v \text{ and } t \end{aligned}$$

and hence

$$Y(t) = \frac{1}{1-\beta} \left(\int_0^{N(t)} x(v,t)^{1-\beta} \right) L^{\beta} = \frac{1}{1-\beta} N(t) L$$

Here $N(t)$ plays the role of technology $A(t)$ in the neoclassical model. If it is expanding at a costant rate then there will be endogenous growth.

Wage are given by the fact that $w(t)L = \beta Y(t)$, hence

$$w(t) = \frac{\beta}{1-\beta} N(t)$$

Free entry into research implies that

$$\eta V(v,t) \leq 1$$

with the "complementary slackness" conditions

$\eta V(v,t) = 1$ if $Z(v,t) > 0$ and $Z(v,t) = 0$ if $\eta V(v,t) < 1$

In a BGP we have $r(t) = r^*$ hence $V^* = \beta L/r^*$ and

$$\eta\beta L/r^* = 1 \Longrightarrow r^* = \eta\beta L$$

The consumption Euler equation then implies that

$$g^* = g_C^* = \frac{1}{\theta}(\eta\beta L - \rho)$$

Need to assume that

$$\eta\beta L > \rho > (1-\theta)(\eta\beta L)$$

where the second inequality comes from the usual $\rho > (1-\theta)g$.

No transition dynamics. Why?

How does the equilibrium solution compare to the optimum solution? What are appropriate policy interventions?