Econ 559
Models of Schumpeterian Growth
Based on Acemoglu, Chapter 14

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Consider a model with

$$Y(t) = \frac{1}{1-\beta} \widetilde{X}(t)^{1-\beta} L^{\beta},$$

but instead of

$$\widetilde{X}(t) = \left(\int_0^{N(t)} x(v,t)^{\frac{\varepsilon-1}{\varepsilon}} dv \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

we now have

$$\widetilde{X}(t) = \left(\int_0^1 \sum_{q \in \Omega(v,t)} q^{\frac{1}{\varepsilon}} x(v,t \mid q)^{\frac{\varepsilon-1}{\varepsilon}} dv \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

Here $\Omega(v,t)$ is the set of qualities q available at time t for variety v, and $x(v,t \mid q)$ is the quantity used of that "machine" variety.

Let $q(v,t) \equiv \max \Omega(v,t)$. If only the higher quality in $\Omega(v,t)$ is used, so that $x(v,t \mid q) = 0$ for $q < q(v,t)$ and $x(v,t \mid q) = x(v,t)$ for $q = q(v,t)$, for all v, then (setting $\varepsilon = 1/\beta$) we have

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^{\beta} x(v,t)^{1-\beta} dv\right) L^{\beta}$$

Note: the book has instead

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^1 q(v,t) x(v,t)^{1-\beta} dv\right) L^{\beta}$$

but the alternative here is more convenient (see below).

Quality ladders:

$$q(v,t) = \lambda^{n(v,t)} q(v,0)$$

Two things to note: (i) Innovation is cumulative; (2) There is room for "creative destruction."

Research is as in the equipment-lab model, so that

$$C(t) + X(t) + Z(t) \leq Y(t)$$

Production of all machines has constant marginal cost ψ and there is instantaneous depreciation, so that

$$\int_0^1 \sum_{q \in \Omega(v,t)} x(v, t \mid q) dv = X(t)/\psi$$

Note: the book has marginal cost scaled up by quality, so that MC is $\psi q(v,t)$. But if this was true, there would be no room for monopoly profits. So the book is mistaken there, in particular regarding equation (14.5).

Research is stochastic: spending $Z(v,t)$ on research on line v, then flow probability of innovation is

$$\eta Z(v,t)/q(v,t)$$

Note: research gets progresively more difficult, but also progresively yields a biger improvement, $\lambda q(v,t)$.

Consider a "rule of thumb allocation" where a share z of $Y(t)$ is devoted to research, while a share x of $Y(t)$ is devoted to $X(t)$:

$$\begin{aligned} Z(t) &= zY(t) \\ X(t) &= xY(t) \end{aligned}$$

Assume also that $x(v, t \mid q) = 0$ for $q < q(v, t)$ and $x(v, t \mid q) = x(t)q(v, t)$ for $q = q(v, t)$, for all v. This implies that

$$\begin{aligned} \int_0^1 \sum_{q \in \Omega(v,t)} x(v, t \quad \mid \quad q) dv = \int_0^1 x(t) q(v, t) dv \\ \implies x(t) = (x/\psi)\, Y(t)/Q(t) \end{aligned}$$

where

$$Q(t) \equiv \int_0^1 q(v, t) dv$$

We then have

$$
\begin{aligned}
Y(t) &= \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^{\beta} x(v,t)^{1-\beta} dv\right) L^{\beta} \\
&= \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^{\beta} x(t)^{1-\beta} q(v,t)^{(1-\beta)} dv\right) L^{\beta} \\
&= \frac{1}{1-\beta}\left(\frac{(x/\psi)\, Y(t)}{Q(t)}\right)^{1-\beta} \left(\int_0^1 q(v,t) dv\right) L^{\beta} \\
&= \frac{1}{1-\beta}(x/\psi)^{1-\beta}\, Y(t)^{1-\beta} Q(t)^{\beta} L^{\beta}
\end{aligned}
$$

and hence

$$
Y(t) = \left(\frac{(x/\psi)^{1-\beta}}{1-\beta}\right)^{1/\beta} Q(t) L
$$

while GDP is

$$
Y(t) - X(t) = (1-x)\left(\frac{(x/\psi)^{1-\beta}}{1-\beta}\right)^{1/\beta} Q(t) L
$$

What is the growth rate of $Q(t)$? Assume that $Z(t)$ is allocated among the v such that $Z(v,t)/q(v,t)$ is the same among v, so that

$$\int \frac{Z(v,t)}{q(v,t)} dv = \frac{Z(t)}{Q(t)} = \frac{zY(t)}{Q(t)}$$

Hence

$$\begin{aligned}
\dot{Q}(t) &= \int (\lambda - 1) \frac{\eta z Y(t)}{Q(t)} q(v,t) dv \\
&= (\lambda - 1) \eta z \left[\frac{(x/\psi)^{1-\beta}}{1-\beta} \right]^{1/\beta} Q(t) L
\end{aligned}$$

so that

$$g = (\lambda - 1) \eta z \left[\frac{(x/\psi)^{1-\beta}}{1-\beta} \right]^{1/\beta} L$$

Things to note:

1. Strong scale effects (see below).

2. The optimal value of x would be set equal to

$$\begin{aligned} x_m &= \max_x (1-x)x^{(1-\beta)/\beta} \\ &\Longrightarrow x_m = 1-\beta \end{aligned}$$

3. The optimal value of z would be set depending on intertemporal preferences.

Alternative models:

1) Using labor for research, with probability of success equal to $\eta L_R(v,t)$ - note, now this flow probability is not affected by $q(v,t)$. Note: think about the hybrid model where the probability of success is $\eta L_R(v,t)^\gamma Z(v,t)^{1-\gamma}$. what do you need to assume to get constant growth?

The rule of thumb is $L_R = zL$ and $L_R(v,t) = L_R$. Output is still

$$Y(t) = \left(\frac{(x/\psi)^{1-\beta}}{1-\beta}\right)^{1/\beta} Q(t)L$$

and now

$$\dot{Q}(t) = \int (\lambda - 1)q(v,t)\eta z L dv = (\lambda - 1)\eta z Q(t)L$$

so that

$$g = (\lambda - 1)\eta z L$$

Again: strong scale effect. Note: no need for spillovers... why?

2) No scale effects (Young, 1998). Go back to first model but with population growth and

$$Y(t) = \frac{1}{1-\beta} N(t)^{-\beta} \left(\int_0^{N(t)} q(v,t) x(v,t)^{1-\beta} dv \right) L(t)^{\beta}$$

with $\dot{N}(t) = L(t)$. This implies that in steady state $L(t)/N(t) = g_N = g_L$, hence $N(t) = L(t)/g_L$.

With rule of thumb being $X(t) = xY(t)$ and $x(v,t \mid q) = x(t)q(v,t)$ for all v, then

$$\begin{aligned} \int_0^{N(t)} x(v,t \quad &| \quad q) dv = \int_0^{N(t)} x(t) q(v,t) dv \\ &\Longrightarrow \quad x(t) = (x/\psi)\, Y(t)/Q(t) \end{aligned}$$

We then have

$$
\begin{aligned}
Y(t) &= \frac{1}{1-\beta}\left(\int_0^{N(t)} q(v,t)^{\beta} x(v,t)^{1-\beta} dv\right)\left(\frac{L(t)}{N(t)}\right)^{\beta} \\
&= \frac{1}{1-\beta}(x/\psi)^{1-\beta} N(t)^{-\beta} Y(t)^{1-\beta} Q(t)^{\beta} L(t)^{\beta}
\end{aligned}
$$

and hence

$$
\begin{aligned}
Y(t) &= \left(\frac{(x/\psi)^{1-\beta}}{1-\beta}\right)^{1/\beta} \frac{Q(t)}{N\ (t)} L(t) \\
&= g_L \left(\frac{(x/\psi)^{1-\beta}}{1-\beta}\right)^{1/\beta} Q(t)
\end{aligned}
$$

What is the growth rate of $Q(t)$? Again, we assume that the flow probability of innovation in v is $\eta Z(v,t)/q(v,t)$. Assume that $Z(t) = zY(t)$ is allocated among the different varieties such that $Z(v,t)/q(v,t)$ is the same among v, so that again we have

$$\int_0^{N(t)} \frac{Z(v,t)}{q(v,t)} dv = \frac{Z(t)}{Q(t)} = \frac{zY(t)}{Q(t)}$$

Hence

$$\begin{aligned}\dot{Q}(t) &= \int_0^{N(t)} (\lambda - 1) \frac{\eta z Y(t)}{Q(t)} q(v,t) dv \\ &= (\lambda - 1) \eta z g_L \left(\frac{(x/\psi)^{1-\beta}}{1-\beta} \right)^{1/\beta} Q(t)\end{aligned}$$

The growth rate is now

$$g = \left((\lambda - 1) \eta z \left(\frac{(x/\psi)^{1-\beta}}{1-\beta} \right)^{1/\beta} - 1 \right) g_L$$

Note: need to assume that $(\lambda - 1)\eta z \left(\frac{(x/\psi)^{1-\beta}}{1-\beta} \right)^{1/\beta} > 1$.

No scale effects, but η and z still matter!

3) Another way to avoid strong scale effects is by following an approach similar to what Jones did for the Romer model. This entail assuming that the flow rate of innovation resulting from an investment of $Z(v,t)$ units of the final good is $\eta Z(v,t)/q(v,t)^{\phi}$, with $\phi > 1$.

Equilibrium Analysis

Only the best vintage is used. Demand is given by

$$x(v,t \mid q) = q(v,t) L p^{x}(v,t \mid q)^{-1/\beta}$$

Monopoly pricing for the best vintage would be

$$p^{x}(v,t) = \frac{\psi}{1-\beta}$$

For this to be an equilibrium we need to be in the "drastic innovation" case, which entails

$$\frac{q(v,t)^{\beta}}{p^{x}(v,t)} > \frac{[q(v,t)/\lambda]^{\beta}}{\psi}$$
$$\implies \lambda > (1-\beta)^{-1/\beta}$$

Assume this condition to hold from here onwards. As exercise, see what happens otherwise.

Assume $\psi = 1 - \beta$ so that all prices are one. Then

$$x(v,t \mid q) = q(v,t) L$$

and profits are given by

$$\pi(v,t) = (1-\psi) q(v,t) L = \beta q(v,t) L$$

The value function $V(v,t)$ satisfies

$$r(t)V(v,t) - \dot{V}(v,t) = \pi(v,t) - z(v,t)V(v,t)$$

where $z(v,t) \equiv \eta Z(v,t)/q(v,t)$ is the flow probability of losing the profit stream due to "creative destruction."

Because of free entry we have

$$\left(\frac{\eta}{\lambda^{-1}q(v,t)}\right)V(v,t) \leq 1$$

so the "complementary slackness" conditions can be written as

$$\eta V(v,t) \leq \lambda^{-1}q(v,t)$$

and

$$\eta V(v,t) = \lambda^{-1}q(v,t) \text{ if } Z(v,t) > 0$$

The Euler condition for consumption is that

$$\dot{C}(t)/C(t) = \frac{1}{\theta}\left(r(t) - \rho\right)$$

In steady state we have that $z(v,t) = z^*$ and

$$V(v,t) = \frac{\beta q(v,t) L}{r^* + z^*}$$

Since we also must have $V(v,t) = q(v,t)/\lambda\eta$ this implies that $\frac{\beta q(v,t)L}{r^*+z^*} = \frac{q(v,t)}{\lambda\eta}$ and hence

$$r^* + z^* = \lambda\eta\beta L$$

But from the Euler condition we then have (using $g_C^* = g^*$) $g^* = \frac{1}{\theta}(r^* - \rho)$ so $r^* = \theta g^* + \rho$, which implies that

$$z^* = \lambda\eta\beta L - \theta g^* - \rho$$

But also we have that $g^* = \dot{Q}/Q = (\lambda - 1)z^*$, and hence

$$\begin{aligned} g^* &= (\lambda - 1)(\lambda\eta\beta L - \theta g^* - \rho) \\ \Longrightarrow & \quad g^* = \frac{\lambda\eta\beta L - \rho}{\theta + (\lambda - 1)^{-1}} \end{aligned}$$

Need

$$\lambda\eta\beta L > \rho > (1-\theta)(\lambda - 1)\eta\beta L$$

Pareto Optimality

Now we have $p^x(v,t) = \psi$, hence

$$x^S(v,t) = q(v,t)\psi^{-1/\beta} L = q(v,t)(1-\beta)^{-1/\beta} L$$

Using

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^\beta x(v,t)^{1-\beta} dv\right) L^\beta$$

this implies

$$\begin{aligned} Y^S(t) &= \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^\beta \left[\frac{q(v,t)L}{(1-\beta)^{1/\beta}}\right]^{1-\beta} dv\right) L^\beta \\ &= (1-\beta)^{-1/\beta} Q^S(t) L \end{aligned}$$

Net output is then

$$\begin{aligned} & Y^S(t) - X^S(t) \\ = \; & (1-\beta)^{-1/\beta} Q^S(t) L - \psi \int_0^1 x^S(v,t) dv \\ = \; & (1-\beta)^{-1/\beta} Q^S(t) L - \psi (1-\beta)^{-1/\beta} L Q^S(t) \\ = \; & \beta (1-\beta)^{-1/\beta} Q^S(t) L \end{aligned}$$

We also have

$$\begin{aligned}\dot{Q}(t) &= \int_0^1 (\lambda - 1) q(v,t) \frac{\eta Z^S(t)}{q(v,t)} dv \\ &= (\lambda - 1)\eta Z^S(t)\end{aligned}$$

Hence the optimal growth problem is

$$\max \int_0^\infty e^{-\rho t} \frac{C^S(t)^{1-\theta} - 1}{1-\theta} dt$$

subject to

$$\dot{Q}(t) = \eta(\lambda - 1)\left[\beta(1-\beta)^{-1/\beta} Q^S(t) L - C^S(t)\right]$$

Following the standard procedure we find

$$g^S = \frac{\dot{C}^S(t)}{C^S(t)} = \frac{1}{\theta}\Big(\eta(\lambda-1)\beta(1-\beta)^{-1/\beta}L-\rho\Big)$$

This compares to the equilibrium growth rate

$$g^* = \frac{\lambda\eta\beta L-\rho}{\theta+(\lambda-1)^{-1}}$$

As $\lambda \rightarrow \infty$ then $g^* \rightarrow \frac{1}{\theta}(\lambda\eta\beta L-\rho)$ whereas g^S increases without bound. This implies that for large λ we have $g^S > g^*$ - equilibrium growth is too low relative to the optimum. But one can also find parameters under which $g^* > g^S$ - equilibrium growth is too high. This is because of the interplay of two opposing forces: appropriability and business stealing effect.

Innovation by Incumbents and Entrants

As above,

$$Y(t) = \frac{1}{1-\beta}\left(\int_0^1 q(v,t)^\beta x(v,t)^{1-\beta} dv\right) L^\beta$$

Innovation available only to incumbents is as above: at cost $z(v,t)q(t,v)$ with flow $\phi z(v,t)$ for $\phi > 0$, with

$$q(v,t) = \lambda^{n(v,t)} q(v,s)$$

where s is the time at which this machine was invented.

Innovation for entrants leads to a new machine of quality $\kappa q(v,t)$ with $\kappa > \lambda$ at flow $\eta(\hat{z}(v,t))/q(v,t)$, where $\eta()$ is decreasing because of external diminishing returns, continuous and differentiable, $z\eta(z)$ is increasing, and the following Inada-type conditions hold

$$\lim_{z \to \infty} \eta(z) = 0 \text{ and } \lim_{z \to 0} \eta(z) = \infty$$

Marginal cost ψ, assume $\psi = 1 - \beta$.

Equilibrium Analysis

As above,

$$x(v,t \mid q) = p^x(v,t \mid q)^{-1/\beta} q(v,t)L$$

Assume that innovations by entrants are drastic, so that entrants can charge the pure monopoly price $\psi/(1-\beta) = 1$ (a fortiori, incumbents charge a price of 1). This requires

$$\begin{aligned} \frac{\kappa^\beta}{p^x(v,t)} &> \frac{1}{\psi} \\ &\Longrightarrow \kappa^\beta > \frac{1}{1-\beta} \\ &\Longrightarrow \kappa > (1-\beta)^{-1/\beta} \end{aligned}$$

Note: I think there is a mistake in the book in equation (14.37).

In an interior BGP we have $z(v,t) = z^*$ and $\hat{z}(v,t) = \hat{z}^*$, with a linear value function: $V(v,t) = \upsilon q(v,t)$. This property comes from the fact that profits and costs are proportional to quality. The linearity of the R&D function for incumbents imply that (if $z^* > 0$) then

$$\begin{aligned} \phi\upsilon(\lambda - 1)q &= q \\ &\implies \upsilon = \frac{1}{\phi(\lambda - 1)} \\ &\implies V(q) = \frac{q}{\phi(\lambda - 1)} \end{aligned}$$

whereas free entry by entrants implies that (if $\hat{z}^*$)

$$\begin{aligned} \eta(\hat{z}^*)V(\kappa q) &= q \\ &\implies V(\kappa q) = \frac{q}{\eta(\hat{z}^*)} \\ &\implies 1 = \frac{\phi(\lambda - 1)}{\kappa\eta(\hat{z}^*)} \end{aligned}$$

This determines $\hat{z}^*$.

Since $\pi(q) = \beta L q$ we also need that

$$V(q) = \frac{\beta L q}{r^* + \hat{z}^*\eta(\hat{z}^*)}$$

Hence

$$\begin{aligned} r^* &= \frac{\beta L q}{q/\phi(\lambda-1)} - \hat{z}^*\eta(\hat{z}^*) \\ &= \phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) \end{aligned}$$

and hence

$$g^* = \frac{1}{\theta}\left(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho\right)$$

But also

$$g^* = \frac{\dot{Q}(t)}{Q(t)} = (\lambda-1)\phi z^* + (\kappa-1)\hat{z}^*\eta(\hat{z}^*)$$

This determines the contribution of incumbent and entrant R&D to growth.

Notice the implication for firm dynamics: in terms of revenue, incumbents grow by λ with probability $\phi z^*\Delta t$, die with probability $\hat{z}^*\eta(\hat{z}^*)\Delta t$, and remain as they are otherwise.