Econ 559
Lecture Notes for The Solow Growth Model
(based on Acemoglu, chapter 2)

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Closed economy, unique final good, discrete time.

Representative household saves fraction $s \in (0, 1)$ of income.

Representative firm with $Y = F(K, L, A)$, role of A (non-rival and non-excludable)

Assumption 1 - continuity, differentiability, positive and diminishing marginal products, and CRS

Definition 2.1: $g(x, y)$ is HDG m in $x, y \in \mathbb{R}$ if

$$g(\lambda x, \lambda y) = \lambda^m g(x, y) \text{ for all } \lambda \in \mathbb{R}_+$$

Theorem 2.1 (Euler's Theorem): If $g(x, y)$ is HDG m in $x, y \in \mathbb{R}$ then

$$mg(x, y) = g_x(x, y)x + g_y(x, y)y \text{ for all } x, y \in \mathbb{R}$$

Moreover, $g_x(x, y)$ and $g_y(x, y)$ are HDG $m-1$ in $x, y \in \mathbb{R}$.

Households own labor and capital and "rent" it to firms.

Inelastic supply of labor. Labor market clearing is then $L(t) = \bar{L}(t)$.

Similarly, $K(t) = \bar{K}(t)$.

Notation: drop the hat and use $L(t)$ and $K(t)$ as supply/endowments.

Depreciation rate δ.

Competitive markets for labor and capital with "rental" prices $w(t)$ and $R(t)$.

Firm's problem is a static (period by period) decision to

$$\max_{K \geq 0, L \geq 0} F(K, L, A(t)) - R(t)K - w(t)L$$

Must have zero profits and

$$\begin{aligned} w(t) &= F_L(K(t), L(t), A(t)) \\ R(t) &= F_K(K(t), L(t), A(t)) \end{aligned}$$

From HDG 1 of F in L and K we then have that

$$Y(t) = w(t)L(t) + R(t)K(t)$$

Inada conditions.

Discrete time dynamics. The two key equations are

$$\begin{aligned} K(t+1) &= (1-\delta)K(t) + I(t) \\ S(t) &= I(t) = Y(t) - C(t) = sY(t) \end{aligned}$$

This leads to the fundamental law of motion of the Solow growth model:

$$K(t+1) = sF(K(t), L(t), A(t) + (1-\delta)K(t)$$

Definition of an equilibrium path: for a given sequence of $\{L(t), A(t)\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ s.t...

Let $k(t) \equiv K(t)/L(t)$, assume that $L(t) = L$ and $A(t) = A$ for all t, and let $f(k, A) \equiv F(K/L, 1, A)$. Then we have

$$k(t+1) = sf(k(t), A) + (1-\delta)k(t).$$

A steady state equilibrium without technological progress and population growth is an equilibrium path with $k(t) = k^*$ for all t.

Figures 2.2, 2.3 and 2.4.

Proposition 2.2: existence and uniqueness with k^* satisfying

$$\frac{f(k^*, A)}{k^*} = \frac{\delta}{s}$$

Proposition 2.3: comparative statics of k^* and $y^* = f(k^*, A)$ w.r.t δ, s, A (note that c is inverted U shaped as function of s, golden rule, dynamic inefficiency).

Continuous time with population growth, $L(t) = L(0)e^{nt}$, $\dot{k}/k = \dot{K}/K - n$ and now the fundamental law of motion of the Solow model in continuous time is

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t), A)}{k(t)} - (n + \delta)$$

Steady state:

$$\frac{f(k^*, A)}{k^*} = \frac{n + \delta}{s}$$

Comparative statics w.r.t. s, A, δ and n.

Transitional dynamics: globally asymptotically stable, monotonic convergence (follows simply from the fact that $f(k, A)/k$ is decreasing in k ($f_k/k - f/k^2 = (1/k^2)(f_k k - f) = -w/k^2 < 0$).

CES technology in intensive form ($A = 1$ and no population growth):

$$y = f(k) = \left(\gamma(Bk)^{(\sigma-1)/\sigma} + 1 - \gamma\right)^{\sigma/(\sigma-1)}$$

Hence

$$\lim_{k\to\infty} f(k)/k = \gamma^{\sigma/(\sigma-1)}B$$

If this is higher than $(n+\delta)/s$ then there is long-run growth (like AK model with $A \equiv \gamma^{\sigma/(\sigma-1)}B$), but some problems...

Balanced growth: K/Y and shares are constant.

How to model technological progress? Different possibilities:

$$\text{Hicks-neutral: } \widetilde{F}(K,L,A) = AF(K,L)$$

$$\text{Capital-augmenting: } \widetilde{F}(K,L,A) = F(AK,L)$$

$$\text{Labor-augmenting: } \widetilde{F}(K,L,A) = F(K,AL)$$

Uzawa's Theorem I: technological progress must be "labor augmenting" for there to be a balanced growth path after $t \geq T$. More formally, if Y and K and C grow at positive rates g_Y, g_K and g_C with $\dot{L}/L = n$ for all $t \geq T$ then (a) $g_Y = g_K = g_C$ and (b) $Y = F(K, AL)$ with $\dot{A}/A = g_Y - n$.

Worth going through the proof. First part: by hypothesis for $t \geq T$ we have $Y(t) = e^{g_Y(t-T)}Y(T)$ and $K(t) = e^{g_K(t-T)}K(T)$ and $L(t) = e^{n(t-T)}L(T)$. Since $\dot{K} = g_K K$, the aggregate resource constraint at time t implies

$$(g_K + \delta)K(t) = Y(t) - C(t)$$

Dividing both sides by $e^{g_K(t-T)}$,

$$(g_K+\delta)K(T) = e^{(g_Y-g_K)(t-T)}Y(T) - e^{(g_C-g_K)(t-T)}C(T)$$

for all $t \geq T$. Differentiating w.r.t. t yields

$$(g_Y-g_K)e^{(g_Y-g_K)(t-T)}Y(T) - (g_C-g_K)e^{(g_C-g_K)(t-T)}C(T)$$

for all $t \geq T$. If K and Y and C are positive, then this implies $g_Y = g_K = g_C$.

Second part: we have

$$Y(T) = \widetilde{F}(K(T), L(T), \widetilde{A}(T))$$

hence using assumption of constant growth rates of Y and K at g_Y and g_K, respectively,

$$e^{-g_Y(t-T)}Y(t) = \widetilde{F}(e^{-g_K(t-T)}K(t), e^{-n(t-T)}L(t), \widetilde{A}(T))$$

hence

$$\begin{aligned}
Y(t) &= \widetilde{F}(e^{(g_Y-g_K)(t-T)}K(t), e^{(g_Y-n)(t-T)}L(t), \widetilde{A}(T)) \\
Y(t) &= \widetilde{F}(K(t), e^{(g_Y-n)(t-T)}L(t), \widetilde{A}(T)) \\
Y(t) &= F(K(t), A(t)L(t)) \text{ with } \dot{A}/A = g_Y - n
\end{aligned}$$

Uzawa's Theorem II: along a balanced growth path (competitive market) factor prices are also given by $w = AF_L(K, AL)$ and $R = F_K(K, AL)$.

Further intuition: $F(A_K K, A_L L)$ with CRS then $A_K K$ and $A_L L$ must grow at a common rate for factor shares to be constant. But this rate would then be $g_Y = g_K$ and hence A_K is constant and $g_K = g_Y = n + g_A$.

Technological progress, $Y = F(K, AL)$ with $g_A = g > 0$. Letting $k = K/AL$, then

$$\dot{k}/k = sf(k)/k - (\delta + g + n)$$

In a BGP (steady state?) k is constant but $Y/L = F(K, AL) = Af(k)$ and $K/L = Ak$ grow at rate g. Asymptotic stability.