Econ 559
Lecture Notes for the Solow Growth Model vs Data
Based on Acemoglu, Chapter 3

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Growth accounting: $Y = F(K, L, A)$, letting $x \equiv \frac{F_A A}{Y} \frac{\dot{A}}{A}$ and $\varepsilon_k \equiv F_K K/Y$ and $\varepsilon_l = F_L L/Y$, $g = \dot{Y}/Y$, $g_K = \dot{K}/K$ and $g_L = \dot{L}/L$,

$$x = g - \varepsilon_k g_K - \varepsilon_l g_L$$

Wih competitive labor markets we have $\varepsilon_k = \alpha_K \equiv RK/Y$ and $\varepsilon_l = \alpha_L \equiv wL/Y$, hence

$$x = g - \alpha_K g_K - \alpha_L g_L$$

For estimation, use averages for factor shares. Solow (1957) found that x is high.

Empirical problems?

Note that in theory, along a BGP, we would have that $g = g_K = g_A + g_L$, hence

$$\begin{aligned} x &= (1 - \alpha_K)\, g - \alpha_L g_L \\ &= (1 - \alpha_K)\, (g_A + g_L) - \alpha_L g_L \\ &= (1 - \alpha_K)\, g_A \end{aligned}$$

The term $\alpha_k g_A$ is a contribution of capital to growth. Alternatively, one could include the term $\alpha_k g_A$ as part of the contribution of technological progress.

Application: Young (The Tyranny of Numbers, QJE 1995) calculates for Hong Kong that $g = 7.3\%$ and $x = 2.3\%$. It would seem that most of output growth comes from inputs rather than TFP. But (1) $g_L = 2.6\%$ hence $g - g_L = 4.7\%$. This is what we really want to explain. And (2) $\alpha_K g_A$ should arguably be attributed to TFP growth rather than to capital. Hence we care about $g_A = x/(1 - \alpha_K)$. With $\alpha_K = 0.38$ for Hong Kong, this implies that $g_A = 3.7\%$. So we go from $x = 2.3\%$ out of $g = 7.3\%$ to $g_A = 3.7\%$ out of $g = 4.7\%$.

The speed of convergence: consider the Solow model with labor-augmenting technological progress, with $y = Y/L$ and $k \equiv K/AL$. Then $Y = F(K, AL)$, $y = AF(K/AL, 1) = Af(k)$. Differentiating w.r.t. time,

$$\frac{\dot{y}(t)}{y(t)} = g + \varepsilon_k(k(t))\frac{\dot{k}(t)}{k(t)}$$

Note two sources of growth. We also have

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (\delta + g + n)$$

First order Taylor expansion w.r.t. $\log k(t)$ around k^* ($g(x) = g(x^*) + g'(x^*)x^* [\log x - \log x^*]$) yields

$$\frac{\dot{k}(t)}{k(t)} \approx (\varepsilon_k(k^*) - 1)(\delta + g + n)(\log k(t) - \log k^*)$$

But similarly

$$\log y(t) - \log y^*(t) \approx \varepsilon_k(k^*)(\log k(t) - \log k^*)$$

so we have

$$\frac{\dot{y}(t)}{y(t)} \approx g - (1 - \varepsilon_k(k^*))(\delta + g + n)(\log y(t) - \log y^*(t))$$

Speed of convergence: $(1 - \varepsilon_k(k^*))(\delta + g + n)$.

Under Cobb-Douglas, have $\varepsilon_k = \alpha = 1/3$ and $\delta = 0.05$, $g = 0.02$, $n = 0.01$, hence the speed of convergence is approximately 0.054.

Regression analysis. Can use the previous equation to write

$$g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t-1}$$

The term b^1 should be negative... problems, hence

$$g_{i,t,t-1} = b_i^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t-1}$$

and think of *conditional* convergence. This is implemented as

$$g_{i,t,t-1} = \mathbf{X}_{i,t}^T \boldsymbol{\beta} + b^1 \log y_{i,t-1} + \varepsilon_{i,t-1}$$

Results imply $b^1 \approx 0.02$. For this to be consistent with the speed of congerence of $(1-\alpha)(\delta + g + n) = (1-\alpha)0.08$ need $1 - \alpha = 0.02/0.08 = 1/4$, or $\alpha = 3/4$.

Adding human capital (Mankiw, Romer and Weil, 1992). $Y = F(K, H, AL)$, Assumptions 1 and 2, $\dot{L}/L = n$, $\dot{A}/A = g$, $k = K/AL$ and $h = H/AL$, $y = Af(k, h)$, savings rates s_k and s_h, depreciation rates δ_k and δ_h, and laws of motion

$$\begin{aligned} \dot{k} &= s_k f(k, h) - (\delta_k + g + n)k \\ \dot{h} &= s_h f(k, h) - (\delta_h + g + n)h \end{aligned}$$

Steady state (existence and uniqueness), stability, BGP with $g_K = g_H = g_Y = g_C = g + n$.

It is interesting to think about convergence in this model. Imagine to simplify that $A = L = 1$ and $y = f(k, h) = k^{\alpha} h^{\beta}$ with $\alpha + \beta < 1$ and $\delta_k = \delta_h = \delta$. Then $\dot{k}/k = s_k k^{\alpha-1} h^{\beta} - \delta$ and similarly for $\dot{h}/h$. Let's log-linearize the system around the steady state (k^*, h^*). We have

$$\begin{aligned}
\frac{\dot{k}}{k} &= (\alpha - 1)\delta(\log k - \log k^*) + \beta\delta(\log h - \log h^*) \\
\frac{\dot{h}}{h} &= (\beta - 1)\delta(\log h - \log h^*) + \alpha\delta(\log k - \log k^*)
\end{aligned}$$

while

$$\begin{aligned}
\frac{\dot{y}}{y} &= \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{h}}{h} \\
&= \delta\left[\alpha + \beta - 1\right]\left[\alpha(\log k - \log k^*) + \beta(\log h - \log h^*)\right] \\
&= \delta\left[\alpha + \beta - 1\right](\log y - \log y^*)
\end{aligned}$$

This implies that the rate of conditional convergence now is $\delta(1 - \alpha - \beta)$. So with $\beta > 0$ we will have a lower rate of conditional convergence that can be closer to the one "found" in the data.

With Cobb-Douglas technology $Y = K^{\alpha}H^{\beta}(AL)^{1-\alpha-\beta}$ and $A_j(t) = \bar{A}_j e^{gt}$, then have

$$\begin{aligned} \log y_j^*(t) &= \log \bar{A}_j + gt \\ &\quad + \frac{\alpha}{1-\alpha-\beta} \log\left(\frac{s_{k,j}}{n_j + g + \delta_k}\right) \\ &\quad + \frac{\beta}{1-\alpha-\beta} \log\left(\frac{s_{h,j}}{n_j + g + \delta_h}\right) \end{aligned}$$

Orthogonal technology assumption: $\bar{A}_j = \varepsilon_j A$, with ε_j orthogonal to all other variables. Then MRW use OLS for

$$\begin{aligned} \log y_j^*(t) &= constant + \frac{\alpha}{1-\alpha-\beta} \log\left(s_{k,j}\right) \\ &\quad - \frac{\alpha}{1-\alpha-\beta} \log\left(n_j + g + \delta_k\right) \\ &\quad + \frac{\beta}{1-\alpha-\beta} \log\left(s_{h,j}\right) \\ &\quad - \frac{\beta}{1-\alpha-\beta} \log\left(n_j + g + \delta_h\right) + \varepsilon_j \end{aligned}$$

Get R^2 around 0.78 with α close to 1/3, however updated data leads to worse results.

Implied $\beta = 0.28$. How can we gauge whether this is reasonable? Implication is that a country with $s_h = 12\%$ would have about 11 times higher income than one with $s_h = 0.04\%$, all else equal. This seems too large based on microeconometric evidence, where productivity can be approximated by the wage, where Mincer regression entails $\log w \approx \phi s$, $\phi \in [0.06, 0.1]$, so productivity of society with $s = 12$ would be between 2 and 3 times higher than one with $s = 0$.

Calibration with Cobb-Douglas, $\alpha = 1/3$, $Y = AK^{\alpha} H^{1-\alpha}$, $H = hL$, $h = \exp^{\phi s}$. One could do some decomposition with $y = A\left(K/L\right)^{\alpha} h^{1-\alpha}$, but note that in steady state $r + \delta = R = A\alpha(K/hL)^{\alpha-1}$ implies

$$K/L = \left(\alpha A/R\right)^{1/(1-\alpha)} h$$

So part of K/L variation is "caused" by A variation and also by h.

A better decomposition takes this endogeneity of K/L into account by using K/Y instead of K/L – note that $K/Y = \alpha/R$ is not affected by A. Klenow-RC and Hall and Jones use

$$
\begin{aligned}
Y &= AK^{\alpha}(hL)^{1-\alpha} \\
1 &= A(K/Y)^{\alpha}(h/y)^{1-\alpha} \\
y &= A^{1/(1-\alpha)}(K/Y)^{\alpha/(1-\alpha)}h = A'X
\end{aligned}
$$

This makes sense because K/Y is independent of A... think about balanced growth. Note amplification of A on y through capital accumulation.

Can differences in X explain large differences in y across countries? Measure X by measuring K (accumulating data on I, using some value for δ) and $h = e^{\phi s}$. Then $A' = y/X$.

How much of the cross-country variation in y is "accounted for" by the variation in X and how much by the variation in A? Starting from $y = A'X$, then $\ln y = \ln A' + \ln X$, so

$$\begin{aligned} \frac{var \ln y}{var \ln y} &= \frac{cov(\ln y, \ln A' + \ln X)}{var \ln y} \\ &= \frac{cov(\ln y, \ln A') + cov(\ln y, \ln X)}{var \ln y} \end{aligned}$$

hence

$$1 = \frac{cov(\ln y, \ln A')}{var \ln y} + \frac{cov(\ln y, \ln X)}{var \ln y} = \widehat{\beta}_{A'} + \widehat{\beta}_X$$

where $\widehat{\beta}_{A'}$ ($\widehat{\beta}_X$) is the coefficient of the OLS regression of $\ln A'$ on $\ln y$ ($\ln X$)

Note: this is simply a variance decomposition when the covariance between $\ln A$ and $\ln k$ is shared equally between A' and X,

$$var \ln y = var \ln A' + var \ln X + 2cov(\ln A', \ln X)$$

$$1 = \frac{var \ln A' + cov(\ln A', \ln X)}{var \ln y} + \frac{var \ln k + cov(\ln A', \ln X)}{var \ln y}$$

$$1 = \frac{cov(\ln y, \ln A')}{var \ln y} + \frac{cov(\ln y, \ln X)}{var \ln y}$$

Results: around 60% of income variation comes from A' - quite different from MRW.

Recent developments:

- role of inputs with an input-output matrix (Jones, 2009).

- endogenous quality of schooling (Manuelli and Sheshadri, Erosa and Restuccia),

- endogenous A (Klenow and RC, Cordoba and Ripoll).

Issue is still up in the air, but TFP differences plus amplification seem important.