Econ 559
The Neoclassical Growth Model
Based on Acemoglu, Chapter 8

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In general, one cannot represent the aggregate excess demand function $x(p)$ as the solution of the maximization problem of a single (representative) household. But this is possible with "Gorman preferences", where the indirect utility function have the form

$$v^h(p, w^h) = a^h(p) + b(p)w^h$$

In this case can use a representative household with preferences

$$v(p, w) = a(p) + b(p)w^h$$

where

$$a(p) \equiv \int a^h(p)dh \text{ and } w \equiv \int w^h dh$$

Moreover, under these preferences there exists a normative representative household: with $N < \infty$ and a convex aggregate production possibilities set Y, a feasible allocation that maximizes utility of representative household is Pareto optimal.

Consider an infinite-horizon economy in continuous time and suppose economy admits normative representative household with instantaneous utility $u(c(t))$.

Assumption 3: neoclassical preferences.

Without loss of generality, assume that there is a continuum of households of measure 1, with each household supplying $L(t)$ inelastically and $L(0) = 1$ with $L(t) = \exp(nt)$.

Utility function is

$$\int_0^\infty e^{-(\rho-n)t} u(c(t))dt$$

with $\rho > n$.

Production represented by $Y = F(K, L)$ satisfying Assumptions 1 and 2 (of Chapter 2). Will use $y = f(k)$, with $k \equiv K/L$, and have

$$\begin{aligned} R &= F_K(K, L) = f'(k) \\ w &= F_L(K, L) = f(k) - kf'(k) \end{aligned}$$

Capital accumulation together with population growth at n implies

$$\frac{\dot{k}}{k} = \frac{f(k) - \delta k - c}{k} - n$$

Law of motion of per-capita assets is

$$\dot{a} = (r - n)a + w - c$$

With no government sector, have $a(t) = k(t)$ for all t, and $r = R - \delta$.

The no-Madoff condition takes the form

$$\lim_{t \to \infty} \left[a(t) \exp\left(-\int_0^t (r(s) - n) ds \right) \right] \geq 0$$

A competitive equilibrium of the NGM consists of

$$[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$$

s.t. representative household maximizes its utility given $K(0) > 0$ and taking $[w(t), R(t)]_{t=0}^{\infty}$ as given; firms maximize profits taking $[w(t), R(t)]_{t=0}^{\infty}$ as given; all factor markets clear.

More convenient to incorporate equilibrium behaviour of firms inside the definition, so that $[w(t), R(t)]_{t=0}^{\infty}$ satisfies equations

$$\begin{aligned} R &= F_K(K, L) = f'(k) \\ w &= F_L(K, L) = f(k) - kf'(k) \end{aligned}$$

Set up Hamiltonian to get necessary conditions and impose transversality condition $\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)a(t) = 0$ (where $\mu(t)$ is the costate variable). Play with the F.O.C. of the maximization of the Hamiltonian w.r.t. $c(t)$ and the costate equation to get the continuous-time consumption Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho)$$

where

$$\varepsilon_u(c) = -\frac{u''(c)c}{u'(c)}$$

is the elasticity of the marginal utility $u'(c)$. From the costate equation and the transversality condition we get

$$\lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s)-n)ds\right)\right] = 0$$

This is the no-Ponzi condition with equality (role of transversality condition).

Any pair $(k(t), c(t))$ that satisfies this equation (with $a(t) = k(t)$) and the Euler equation corresponds to a CE. The full CE is this combined with market clearing prices, which entail $r(t) = f'(k(t)) - \delta$. Substituting $r(t) = f'(k(t)) - \delta$ into the Euler equation we get

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho)$$

and into the transversality condition (with $a(t) = k(t)$) yields

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (f'(k(s)) - \delta - n)ds\right)\right] = 0$$

Together with

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$$

these three equations completely determine the CE.

How do you go about proving optimality in this case?

Steady state: $\dot{c} = 0$, hence $f'(k^*) = \rho + \delta$. Note that k^* (and thus $c^* = f(k^*) - (n + \delta)k^*$) is independent of $u()$ function. The shape of $u(c)$ affects the transition but not the steady state. Why?

Compare to k^*_{gold} defined by $f'(k^*_{gold}) = n + \delta$.

In the Solow model we had $\partial k^*/\partial n < 0$, whereas now $\partial k^*/\partial n = 0$. Why?

Transitional dynamics are determined by the system of two differential equations

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho)$$

and

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$$

together with $k(0)$ and the boundary condition

$$\lim_{t \to \infty} \left[k(t) \exp \left(- \int_0^t (f'(k(s)) - \delta - n) ds \right) \right] = 0$$

The system exhibits the existence and uniqueness of an equilibrium with (saddle-path) stability. Figure 8.1.

Now allow for exogenous, constant, labor augmenting technological progress, so that $Y(t) = F(K(t), A(t)L(t))$, with $A(t) = A(0)e^{gt}$.

We now have $y \equiv Y/L = Af(k)$, where now $k \equiv K/AL$.

Is there a balanced growth path? The Euler equation is (as above)

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho)$$

In a BGP we have $r(t) = r^*$ and $\dot{c}/c = n+g$ implies that we need ε_u (defined above as the elasticity of the marginal utility $u'(c)$ and given by $\varepsilon_u(c) = -u''(c)c/u'(c)$) to be constant.

$\varepsilon_u(c)$ is also the Arrow-Pratt coefficient of relative risk aversion and also the limit as $s \to t$ of the intertemporal elasticity of subsitution in consumption between times s and t, defined as

$$\sigma_u(t,s) = -\frac{d \log(c(s)/c(t))}{d \log(u'(c(s))/u'(c(t)))}$$

Integrating $\theta = -u''(c)c/u'(c)$ yields the family of CRRA utility functions as

$$u(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \log c & \text{if } \theta = 1 \end{cases}$$

It turns out that the CRRA utility function is the only separable utility function in the family of Gorman preferences.

Canonical model: labor augmenting constant exogenous technological change and CRRA preferences,

$$\int_0^\infty e^{-(\rho-n)t}\frac{c(t)^{1-\theta}-1}{1-\theta}dt$$

Let $\widetilde{c} \equiv c/A$. Then

$$\frac{d\widetilde{c}/dt}{\widetilde{c}} = \frac{\dot{c}}{c} - g = \frac{1}{\theta}\left(r(t) - \rho - \theta g\right)$$

We also have

$$\begin{aligned}
\dot{K} &= Af(k)L - cL - \delta K \\
&\rightarrow \dot{k} = \frac{\dot{K}}{AL} - \frac{\left(\dot{A}L + A\dot{L}\right)K}{(AL)^2} \\
&= \frac{1}{AL}\left(Af(k)L - cL - \delta K\right) - \left(\frac{\dot{A}}{A} - \frac{\dot{L}}{L}\right)\frac{K}{AL} \\
&= f(k) - \widetilde{c} - (\delta + g + n)k
\end{aligned}$$

The transversality condition is now

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t (f'(k(s))-\delta-n-g)ds\right)\right]=0$$

Since $r(t) = R(t) - \delta = f'(k) - \delta$ and $\widetilde{c}$ must be constant along a BGP then $r(t) - \rho - \theta g = 0$ and so

$$f'(k^*) = \rho + \delta + \theta g$$

This equation determines k^* while

$$\widetilde{c}^* = f(k^*) - (n + g + \delta)k^*$$

Substituting back into the transversality condition we get the condition that

$$\rho - (1 - \theta)g - n > 0$$

or

$$\rho - n > (1 - \theta)g$$

which is an assumption we must make (analogous to $\rho - n$ before). This guarantees $r^* > g + n$.

Under standard neoclassical assumptions about $F(K, L)$ and CRRA preferences and with $\rho - n > (1 - \theta)g$ then a BGP exists, is unique, and globally stable.

Is the rate of conditional convergence higher or lower in the canonical NGM than in the Solow model? It depends on the behaviour of the savings rate with k/k^*, which in turn depends on θ and the rest of parameters.

For reasonable parameters one gets similar rates of conditional convergence as in the Solow model, so endogenous savings do not help the Solow model better match the data. See Barro and Sala-i-Martin's Economic Growth textbook.

Formally, the dynamic system for $f(k) = Ak^{\alpha}$ is (I change notation so that now I simply use c for $\widetilde{c} = c/A$)

$$\begin{aligned} \dot{k}/k &= Ak^{\alpha-1} - c/k - (g+n+\delta) \\ \frac{\dot{c}}{c} \log c/dt &= \frac{1}{\theta}\left[\alpha A k^{\alpha-1} - (\delta+\rho+\theta g)\right] \end{aligned}$$

Using $z = \log k$ and $x = \log c$ then we can rewrite this as

$$\begin{aligned} \dot{z} &= Ae^{-(1-\alpha)z} - e^{x-z} - (g+n+\delta) \\ \dot{x} &= \frac{1}{\theta}\left[\alpha A e^{-(1-\alpha)z} - (\delta+\rho+\theta g)\right] \end{aligned}$$

The steady state of this system is

$$\begin{aligned} Ae^{-(1-\alpha)z^*} - e^{x^*-z^*} &= a_z \equiv g+n+\delta \\ \alpha A e^{-(1-\alpha)z^*} &= a_x \equiv \delta+\rho+\theta g \end{aligned}$$

Taking a first-order Taylor expansion of this system around the steady-state values yields

$$\begin{aligned}
\dot{z} &= \left[-A(1-\alpha)e^{-(1-\alpha)z^*} + e^{x^*-z^*}\right](z-z^*) \\
&\quad -e^{x^*-z^*}(x-x^*) \\
&= \left[e^{x^*-z^*} - Ae^{-(1-\alpha)z^*} + \alpha Ae^{-(1-\alpha)z^*}\right](z-z^*) \\
&\quad -\left[Ae^{-(1-\alpha)z^*} - a_z\right](x-x^*) \\
&= [a_x - a_z](z-z^*) + [a_z - a_x/\alpha](x-x^*)
\end{aligned}$$

and

$$\begin{aligned}
\dot{x} &= \frac{1}{\theta}\alpha A(1-\alpha)e^{-(1-\alpha)z^*}(z-z^*) \\
&= -\frac{1}{\theta}(1-\alpha)a_x(z-z^*)
\end{aligned}$$

Putting this together in matrix notation,

$$\begin{bmatrix} dz/dt \\ dx/dt \end{bmatrix} = \begin{bmatrix} a_x - a_z & a_z - a_x/\alpha \\ -(1-\alpha)a_x/\theta & 0 \end{bmatrix} \begin{bmatrix} z - z^* \\ x - x^* \end{bmatrix}$$

The determinant of the A matrix is negative as long as $\alpha a_z < a_x$, or which (since $\alpha < 1$) is true as long as $a_z < a_x$, or

$$\begin{aligned} g + n + \delta &< \delta + \rho + \theta g \\ \rho - n &> (1-\theta)g \end{aligned}$$

The negative sign of the determinant means that the two eigenvalues have opposite signs, a result that implies saddle-path stability.

The characteristic equation of the A matrix is

$$\varepsilon^2 - (a_x - a_z)\varepsilon + \gamma = 0$$

where $\gamma \equiv (1-\alpha)a_x\left(a_z - a_x/\alpha\right)/\theta$. The formula for the negative eigenvalue ε_2 is

$$2\varepsilon_2 = (a_x - a_z) - \left[(a_x - a_z)^2 + 4\gamma\right]^{1/2}$$

The solution for z is

$$z = z^* + \psi_1 e^{\varepsilon_1 t} + \psi_2 e^{\varepsilon_2 t}$$

for some constants ψ_1 and ψ_2. Since $\varepsilon_1 > 0$ then $\psi_1 = 0$ given saddle-path stability. Then need $\psi_2 = z(0) - z^*$. Thus, letting $\beta = -\varepsilon_2$,

$$\log k(t) = (1 - e^{-\beta t})\log k^* + e^{-\beta t}\log k(0)$$

and since $\log y(t) = \log A(t) + \alpha \log k(t)$ then

$$\log y(t) = (1 - e^{-\beta t})\log y^* + e^{-\beta t}\log y(0)$$

so β is the rate of convergence. Plugging in the values as before (lecture notes 2, chapter 3) with $\theta = 1$ get $\beta = 0.054$.