Econ 559
Growth with Overlapping Generations
Based on Acemoglu, Chapter 9

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Time is discrete and infinite. Individuals live for two periods. People born at time t have utility given by

$$U_1(c_1(t), c_2(t+1)) = u(c_1(t)) + \beta u(c_2(t+1))$$

Individuals can work only when young and supply labor inelastically, and

$$L(t) = (1+n)^t L(0)$$

Production is $Y = F(K, L)$, so $y = f(k)$ with $k = K/L$

Full depreciation, $\delta = 1$, so $R(t) = f'(k(t))$, and $w(t) = f(k(t)) - k(t)f'(k(t))$

Consumers' problem is

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$\begin{aligned} c_1(t) + s(t) &\leq w(t) \\ c_2(t+1) &\leq [1 + r(t+1)]\, s(t) \end{aligned}$$

with $1 + r(t+1) = R(t+1)$

Note that there is no altruism or bequest motive

The Euler equation is

$$u'(c_1(t)) = \beta R(t+1) u'(c_2(t+1))$$

Together with

$$c_1(t) + \frac{c_2(t+1)}{R(t+1)} = w(t)$$

this defines $c_1(t)$ and $c_2(t+1)$ as a function of $w(t)$ and $R(t+1)$, hence we can write

$$s(t) = s(w(t), R(t+1))$$

$s(w, R)$ is strictly increasing in w but may be increasing or decreasing in R. Why?

The fundamental law of motion for capital here is

$$k(t+1) = \frac{s(w(k(t)), R(k(t+1)))}{1+n}$$

Under **CRRA preferences** ($u(c) = \frac{c^{1-\theta}-1}{1-\theta}$) the Euler equation is

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta}$$

which implies

$$\frac{R(t+1)s(t)}{w(t)-s(t)} = (\beta R(t+1))^{1/\theta}$$

or

$$s(t) = \frac{w(t)}{1+\beta^{-1/\theta}R(t+1)^{1-1/\theta}} \equiv \frac{w(t)}{\psi(t+1)} < w(t)$$

Note that $\theta \lesseqgtr 1$ determines whether $s(t)$ is increasing or decreasing in $R(t+1)$.

Assume **Cobb-Douglas technology**: $f(k) = k^{\alpha}$. From the law of motion

$$k(t+1) = \frac{s(w(k(t)), R(k(t+1)))}{1+n}$$

and $s(t) = w(t)/\psi\,(t+1)$ we get

$$k(t+1) = \frac{1-\alpha}{1+n}\frac{k(t)^{\alpha}}{1+\beta^{-1/\theta}(\alpha k(t+1)^{\alpha-1}))^{-(1-\theta)/\theta}}$$

We can rewrite this as $k' = a\frac{k^{\alpha}}{1+b(k')^{\phi}}$ where a and b are positive constants and $\phi \equiv (1-\alpha)(1-\theta)/\theta$, and then show that

$$\frac{dk'}{dk} = \frac{a^{1/\alpha}\alpha\left(k'(1+b\,(k')^{\phi})\right)^{\frac{\alpha-1}{\alpha}}}{1+(1+\phi)b\,(k')^{\phi}}$$

This shows that dk'/dk is positive, and differentiation reveals (check!) that the RHS is decreasing in k'. This establishes that the function $k'(k)$ is concave, which implies uniqueness and stability.

Note also that the savings rate is

$$\frac{s(t)}{k(t)^{\alpha}} = \frac{1-\alpha}{1+bk(t+1)^{(1-\alpha)(1-\theta)/\theta}}$$

If $\theta < 1$ then $s(t)/k(t)^{\alpha}$ will be decreasing in $k(t+1)$, so convergence will be faster than if $\theta = 1$, where we have a constant savings rate as in the Solow model.

Note: the **canonical OLG model**, which has logarithmic preferences and Cobb-Douglas technologies, has a constant savings rate of $(1-\alpha)\beta/(1+\beta)$, and behaves as the Solow economy in discrete time.

Possibility of **dynamic inefficiency**: let total consumption be $c(t) = c_1(t) + c_2(t)/(1+n)$. But

$$c(t) = f(k(t)) - s(t)$$

so in steady state we have (recall $k(t+1) = s(t)/(1+n)$)

$$c^* = f(k^*) - (1+n)k^*$$

which is maximized for k_{gold} where

$$f'(k_{gold}) = 1 + n$$

Can we have dynamic inefficiency in the OLG model? Yes! Consider the canonical model. Then in steady state we have $(1-\alpha)\beta/(1+\beta)\,(k^*)^\alpha = (1+n)k^*$ and hence

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{1/(1-\alpha)}$$

whereas

$$k_{gold} = \left[\frac{\alpha}{1+n}\right]^{1/(1-\alpha)}$$

If $\frac{\beta(1-\alpha)}{1+\beta} > \alpha$ then $k^* > k_{gold}$. This is satisfied if α is small.

Recall that $r(t) = R(t) - 1 = f'(k(t)) - 1$. Thus, if $k^* > k_{gold}$, then $r^* = f'(k^*) - 1 < f'(k_{gold}) - 1 = n$.

In constrast, in the NGM with no technological change we had (in steady state) $r^* = \rho > n$, where the inequality is needed to satisfy the transversality condition.

If $k^* > k_{gold}$ so that $r^* < n$ then reducing the capital stock by Δk (small) from $t = T + 1$ onwards clearly benefits the people alive at T, while for $t \geq T+1$ we have an increase in consumption by the fact that $k^* > k_{gold}$.

Why doesn't this happen automatically? Because the 1st Welfare Theorem doesn't apply in economies with an infinite number of consumers. Why is there overaccumulation? How can we get rid of it?

Impure altruism

People care about children only indirectly through the "warm glow" of bequests. Assume $n = 1$ and that people work only when adult, and at that time allocate their total income into consumption and bequests to maximize $\log c + \beta \log b$. The young invest their bequests so that total income in adulthood is $w(t) + b(t-1)R(t)$. Clearly

$$b(t) = \frac{\beta}{1+\beta}\left[w(t) + b(t-1)R(t)\right]$$

so

$$k(t+1) = b(t) = \frac{\beta}{1+\beta} f(k(t))$$

With CD technology we have that

$$k^* = \left[\frac{\beta}{1+\beta}\right]^{1/(1-\alpha)}$$

This compares with

$$k^* = \left[\frac{\beta(1-\alpha)}{1+\beta}\right]^{1/(1-\alpha)}$$

in the canonical OLG growth model with $n = 1$. Why the difference?

OLG in continuous time with perpetual youth

Constant (Poisson rate) probability of death v and logarithmic preferences, so utility is

$$\int_0^\infty e^{-(\rho+v)t} \log c(t)dt$$

and

$$\dot{L}(t) = (n - v)L(t)$$

The flow budget constraint is

$$\dot{a}(t \mid \tau) = r(t)a(t \mid \tau) - c(t \mid \tau) + w(t) + z(a(t \mid \tau))$$

where $z(a)$ are annuity payments at time t to an individual with assets of a. Assuming free entry into the "death insurance" market then it must be that $va = z(a)$, hence we can write

$$\dot{a}(t \mid \tau) = (r(t) + v)\, a(t \mid \tau) - c(t \mid \tau) + w(t)$$

Production side as the Solow model in continuous time, only change in law of motion of k is that $g_L = n - v$, hence

$$\dot{k}(t) = f(k(t)) - (n - v + \delta)k(t) - c(t)$$

where $c(t)$ is average consumption.

Given logarithmic preferences it is easy to derive that

$$\frac{\dot{c}(t \mid \tau)}{c(t \mid \tau)} = r(t) - \rho$$

Together with the transversality condition and the law of motion of $a(t \mid \tau)$ this implies

$$c(t \mid \tau) = (\rho + v)\left[a(t \mid \tau) + \omega(t)\right]$$

where

$$\omega(t) = \int_t^\infty e^{-(\bar{r}(s,t)+v)(s-t)} w(s) ds$$

is the present value of the future stream of wages (labor wealth), with

$$\bar{r}(s,t) = \frac{1}{s-t} \int_t^s r(u) du$$

But then

$$c(t) = (\rho + v)\,[a(t) + \omega(t)]$$

and

$$\dot{c}(t) = (\rho + v)\,[\dot{a}(t) + \dot{\omega}(t)]$$

Using

$$\dot{a}(t) = (r(t) - (n - v))a(t) + w(t) - c(t)$$

and

$$(r(t) + v)\,\omega(t) = \dot{\omega}(t) + w(t)$$

yields

$$\dot{c}(t) = (r(t) - \rho)c(t) - (\rho + v)na(t)$$

Intuitively, there is a share n of people at every time that were just born with $a = 0$, so they consume $(\rho + v)a$ less than the average agent.

Using $a(t) = k(t)$ and $r(t) = f'(k(t)) - \delta$ then

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + v)n\frac{k(t)}{c(t)}$$

The new term $((\rho + v)nk(t)/c(t))$ is again related to the newly born who have no assets.

The system is characterized by the previous equation together with the law of motion of $k(t)$,

$$\frac{\dot{k}(t)}{k(t)} = \frac{f(k(t))}{k(t)} - (n - v + \delta) - \frac{c(t)}{k(t)}$$

plus $k(0)$ and the transversality condition.

The steady state is characterized by

$$
\begin{aligned}
f'(k^*) - \delta - \rho - (\rho + v)n\frac{k^*}{c^*} &= 0 \\
\frac{f(k(t))}{k(t)} - (n - v + \delta) - \frac{c(t)}{k(t)} &= 0
\end{aligned}
$$

The phase diagram shows that the system is saddle-path stable and that $k^* < k_{mgr} < k_{gold}$, so that we have dynamic efficiency.

Note: recall that k_{mgr} is the modified golden rule, giving us the k at the steady state in the NGM, where $f'(k_{mgr}) = \delta + \rho$, while k_{gold} is defined by $f'(k_{gold}) = \delta + n - v$).

Why is there dynamic efficiency here?

In the NGM (infinite lives) in steady state since $r_{mgr} = \rho$ then people have flat consumption paths and all households keep assets of $a = k_{mgr}$.

This couldn't work in the OLG model because flat consumption paths imply no asset accumulation, and with "people turnover" we would have a and k go to zero. So we need $r^* > r_{mgr} = \rho > r_{gold} = n - v$.

Blanchard (1985) shows that if labor units decline with "age" (so that effective labor units at time t for somebody born at time τ are $\exp(-\xi(t-\tau))$) then dynamic inefficiency is possible, even with $v = 0$.

To understand this, consider the discrete time, two period lived, OLG model with no discounting ($\beta = 1$) and no capital, with labor "endowments" of l_1 and l_2 when young and old, and production function $y = l$. If $u'' < 0$ (strict concavity, which is assumed in Acemoglu's assumption 3, chapter 8) then there is dynamic efficiency under symmetric labor endowments if $l_1 = l_2$, but if $l_1 > l_2$ then there is dynamic inefficiency.