FRAGILE COALITIONS
UNDER SOCIAL AND SAVINGS FRICTIONS

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Question

- Risk sharing with limited commitment
  - Optimal insurance contract against idiosyncratic shock subject to limited enforcement
  - Application: village insurance, consumption inequality, sovereign debt contract

- Typical assumption: individual deviation & no saving technology

- Exceptions:
  - Krueger and Uhlig (2006): savings via financial intermediaries
  - Genicot and Ray (2003): group deviations
Question: How does the possibility of group deviations affect risk sharing? Does the ability to save interact with this possibility?

- We examine effects of social efficiency and financial efficiency with group deviations.

Related work: Genicot and Ray (2003, Restud)

- Consider impact of group size with sub-group deviations
- Assume deviating coalition come only from original group.
- Assume no savings
- Look only at Markov arrangements.

Kreuger and Uhlig: special case of our analysis
Social Efficiency and Saving Efficiency

Groups of Individuals form Coalitions to insure against risk.

[Social Efficiency]

- There can be barriers to formation.
- Social Efficiency = probability of successfully forming coalition
  - Social efficiency $\uparrow \implies$ Easier to form a coalition
  - Social efficiency $\uparrow \implies$ Easier to form an alternative coalition too
- How does greater social efficiency affect outcomes?

[Savings Efficiency]

- Societies can use storage to bond the coalition.
  - But alternative coalition can also use storage
- Savings Efficiency = return to storage
- How does greater savings efficiency affect outcomes?
Efficient risk-sharing allocation can be characterized by
- consumption floors
- decay rates
- public saving

Social efficiency ($\pi$) $\uparrow \Rightarrow$ (Ex ante) Risk sharing $\sim$ (hump shape)

Savings efficiency ($R$) $\uparrow \Rightarrow$
\[
\begin{cases}
\text{risk sharing}$\downarrow$ (at low $R$) \\
\text{risk-sharing}$\uparrow$ (at high $R$)
\end{cases}
\]

Interaction between $\pi$ and $R$:
- substitutes when the social efficiency ($\pi$) is very low
- complements at higher level of social efficiency ($\pi$).
ECONOMY

Classic Insurance Economy

- Risk averse individuals subject to income risk.

\[
E \left\{ \sum_{t=1}^{\infty} u(c_t) \right\} \quad u \text{ is CRRA; } \gamma
\]

\[y_t \in Y = \{y_1, \ldots, y_N\} \text{ i.i.d. across time and people}\]

- Population is infinite.

- Period 0 is planning/coalition

- Income begins in period 1.
Insurance Coalitions

- Coalition formation (Initial and Deviating):
  - Initial coalition: formed with probability \( \pi \)
  - Deviating Coalition:
    - Can exit from old coalition to form new one with probability \( \pi \).
    - Can include in new coalition people not in original one.
  - If (initial or deviating) coalition does not form then stuck in autarky forever.

- Saving (storage):
  - Gross interest rate of saving: \( R \in [0, \beta^{-1}] \).
  - Saving can be done both on own or within group.
  - If by person, can take it when leave. By coalition cannot.
Optimal Allocation of Coalition

- Optimal allocation solves the social planning problem.
  - Maximize (utilitarian) social welfare of the coalition subject to RC and PC
  - Allocation of deviating coalition also solves the social planning problem.

- Individual weights in the planing problem:
  - Initial coalition: Ex ante identical \(\Rightarrow\) equal weight
  - Deviating coalition: equal weight
    - In first period of deviating: cannot exit or undo.
    - From the second period: identical
Plan of Talk

1. Basics of Coalition and Storage Usage
   - size of coalition, usage of storage

2. First Best Analysis

3. Optimal Coalition allocation characterization

4. Effects of $\pi$ and $R$
   - Is higher $\pi$ good?
   - Is higher $R$ good?
   - Are $\pi$ and $R$ substitutes or complements?
Coalition outcome is \( \{ c_i (y^{N,t}) , s_i (y^{N,t}) , S (y^{N,t}) \} \)

- \( N \) is the number of members
- \( y^{N,t} \) is the history of the vector of income realizations
- \( c_i \) and \( s_i \) are the consumption and savings of \( i \)
- \( S_t \) are the coalition level savings.
**Basics of Coalitions**

**Proposition**

Infinite coalitions (i.e. $N = \infty$) are always optimal.

$\Rightarrow$ No aggregate risk

**Proposition**

Breakaway Coalitions will be homogeneous w.r.t. initial income.

$\Rightarrow$ Deviating coalitions start with $y$ - homogenous initial income.
Basics of Storage

- Individual level storage? No.

**Proposition**

*Storage within a coalition will only take place at level of the coalition (i.e. $s_i (y^N_t) = 0$)*

$\Rightarrow$ Coalition outcome is $\{c(y^t), S_t\}$

- Storage never be used in any complete efficient arrangement.

\[ u'(E\{Y\}) \leq \beta Ru'(E\{Y\}). \]

- Storage will not be used in autarky storage if

\[ u'(\max\{y \in Y\}) \geq \beta RE \{u'(y_i)\}, \quad (N \ ST \ AUT) \]
First Best

Assumption

Assume for now condition \((N\ ST\ AUT)\) holds.

\[ \implies \text{No storage in autarky} \]

\[ \implies \text{Autarky payoff:} \]

\[ V^A(y) = u(y) + \beta V^A, \text{ where } V^A = E\{u(y_i)\}/(1 - \beta) \]

- First Best Ex Ante payoff:
  \[ \pi \tilde{V} + (1 - \pi) V^A, \text{ where } \tilde{V} = u(E\{Y\})/(1 - \beta). \quad (V\ FB) \]

- Payoff from deviating:
  \[ u(y) + \beta \left[ \pi \tilde{V} + (1 - \pi) V^A \right]. \]
For FB to be feasible,

\[
\frac{u(E\{Y\})}{1 - \beta} \geq u(y) + \frac{\beta [\pi u(E\{y\}) + (1 - \pi)E\{u(y)\}]}{1 - \beta}.
\] (FB fes)

When FB is possible at \(\pi = 0\),

- Payoff conditional on initial coalition formation is weakly declining in \(\pi\).
- Ex ante payoff is
  - strictly increasing in \(\pi\) at least up until FB is feasible
  - may or may not be increasing thereafter.
First Best

- Effects of Storage:
  - Increasing $R$ up until $\beta^{-1}$ no direct impact on FB (not used).
  - But it will raise $V^A(y)$. The FB can break down at lower $\pi$.

Proposition

If FB possible with $R = 0$, then increasing $R$ is always weakly bad. If the cut-off level of $\pi$ at which storage breaks down is interior, i.e.

$$\frac{u(E \{ Y \})}{1 - \beta} = \pi \left[ u(y) + \frac{\beta u(E \{ Y \})}{1 - \beta} \right] + (1 - \pi)V^A(y; R) \text{ for } \pi \in (0, 1),$$

this cut-off is strictly decreasing in $R$ when (N ST AUT) does not hold and storage is used in autarky.
Characterization of Allocation:

Roadmap

1. Consider outcomes when \( R = 0 \), and storage is never used.
   
   1.1 Develop approximation to optimal arrangement
   1.2 Characterize how outcomes depend upon \( \pi \).

2. Consider outcomes when \( R \) is big enough so always used.
   
   2.1 Show approximation is exact here.
   2.2 Characterize how outcomes depend upon \( R \) and \( \pi \).

3. Consider outcomes when \( R \) may be used but not always.
   
   3.1 Especially possible in deviating coalitions.
   3.2 Also possible if storage starts to be used in transition.
   3.3 Extend approximation algorithm and characterize.
1. \( R = 0 : \text{Coalition Problem} \)

\[
F(\bar{V}, \pi) = \max_{c(y^t)} E \left\{ \sum_{t=1}^{\infty} \beta^t u(c_t(y^t)) \Pr(y^t) \right\}
\]

subject to

\[
\sum_{y^t} c_t(y^t) \Pr(y^t) = Y
\]

\[
\beta^t u(c_i(y^t)) + E \left\{ \sum_{n=t+1}^{\infty} \beta^k u(c(y^n)|y^t) \right\} \\
\geq \beta^t u(y_t(y^t)) + \beta^{t+1} \left[ \pi \bar{V} + (1 - \pi)V^A \right]
\]

for all \( y^t \),

(1)
1. \( R = 0 : \) Coalition Problem

**Proposition**

- The optimal continuation payoff \( \bar{V} \in \mathcal{V} = \left[ V^A, u(E\{Y\})/(1 - \beta) \right] \).
- The operator \( F : \mathcal{V} \times [0, 1] \to \mathcal{V} \) defines the unique conditional payoff to our contracting problem, conditional the value of \( \bar{V} \) in constraint (1).
- \( F \) is continuous and decreasing in \( \bar{V} \) and \( \pi \)
- The optimum \( \bar{V}^* \) is a fixed point of \( F \), or

\[
\bar{V}^* = F(\bar{V}^*, \pi).
\]
1. $R = 0$: Characterizing Lagrangian

\[
\mathcal{L} = \max_{\{c_t\}} \min_{\{\omega_t, \gamma_t\}} \sum_{t=1}^{\infty} \beta^t \sum_{y^t} \left\{ \left[ 1 + \sum_{n=1}^{t} \omega_n (y^n(y^t)) \right] u(c_t(y^t)) \right\} \Pr(y^t) \\
-\mathbb{E} \sum_{t=1}^{\infty} \beta^t \sum_{y^t} \omega_t(y^t) \{ u(y_t(y^t)) + \beta [\pi \bar{V} + (1 - \pi)V^A] \} ,
\]

The f.o.c. for consumption:

\[
\omega_t(y^t)u'(c_t(y^t)) = \gamma_t.
\]

\[
\implies c_t(y^t) = u'^{-1} \left( \frac{\gamma_t}{\omega_t(y^t)} \right).
\]
1. \( R = 0 \) : Characterizing Lagrangian

Since utility is CRRA with coefficient of risk aversion \( \gamma \):

\[
c_t(y^t) = \left( \frac{\gamma_t}{w_t(y^t)} \right)^{-1/\gamma}.
\]

Resource constraint:

\[
Y = \sum_{y^t} c_t(y^t) \Pr(y^t) = \gamma_t^{-1/\gamma} \sum_{y^t} w_t(y^t)^{1/\gamma} \Pr(y^t).
\]

\[\implies\] Shadow price of consumption:

\[
\gamma_t = \left[ \frac{Y}{\sum_{y^t} w_t(y^t)^{1/\gamma} \Pr(y^t)} \right]^{-\gamma},
\]

\[\implies\] growth rate of shadow price:

\[
\frac{\gamma_t}{\gamma_{t-1}} = \left[ \frac{\sum_{y^{t-1}} w_{t-1}(y^{t-1})^{1/\gamma} \Pr(y^{t-1})}{\sum_{y^{t-1}} w_{t-1}(y^{t-1})^{1/\gamma} \Pr(y^{t-1})} \right]^\gamma = g_t^\gamma.
\]
1. $R = 0$ : Optimal Allocation

- If the participation constraint does not bind,

$$c_t(y^t) = \left[ \frac{\sum_{y_{t-1}} w_{t-1}(y^{t-1})^{1/\gamma} \Pr(y^{t-1})}{\sum_{y_t} w_t(y^t)^{1/\gamma} \Pr(y^t)} \right] c_{t-1}(y^{t-1}) = \frac{c_{t-1}(y^{t-1})}{g_t}$$

**Proposition**

The efficient consumption allocation is determined by a sequence of consumption floors, $\{\bar{c}_t(y_1), ..., \bar{c}_t(y_N)\}$, and decay rates, $\{g_t\}$, where

$$c_t(y^t) = \begin{cases} 
  c_{t-1}(y^{t-1})/g_t & \text{if } c_{t-1}(y^{t-1})/g_t \geq \bar{c}_t(y_t) \\
  \bar{c}_t(y_t) & \text{o.w.}
\end{cases}$$

**Assumption**

Consumption allocation eventually becomes stationary, then

$$\bar{c}_t(y_i) \to \bar{c}(y_i) \text{ and } g_t \to g.$$
2. Adding Storage

- Autarky value with storage:

\[ V^A(y; R) = \max u(c_1) + E \left\{ \sum_{t=2}^{\infty} \beta^{t-1} u(c_t) \right\} \text{ subject to } \]

\[ y_t + R s_{t-1} = s_t + c_t \]

with

\[ y_1 = y, \ s_0 = 0, \ \text{and } s_t \geq 0 \text{ for all } t \geq 1. \]

- The outside option payoff:

\[ \pi \bar{V}(y) + (1 - \pi) V^A(y). \]

\[ \implies \text{depends upon } y \text{ through both Autarky and the deviating coalition} \]
2. Adding Storage

- resource constraint:

\[
\sum_{y^t} c_t(y^t) \Pr(y^t) = Y + RS_{t-1} - S_t,
\]

\[S_0 = 0, \ S_t \geq 0 \text{ for all } t \geq 1.\]

- f.o.c. for \(S_t\):

\[-\beta^t \gamma_t + \beta^{t+1} R \gamma_{t+1} \leq 0 \text{ and } = \text{ if } S_t > 0.\]

\[\implies g_{t+1}^\gamma = \frac{\gamma_{t+1}}{\gamma_t} \leq \frac{1}{\beta R} \text{ and } = \text{ if } S_t > 0.\]

- Implication:

**Proposition**

*Storage is used in the mechanism iff it can help to smooth the shadow price of consumption to planner.*

\[\implies \text{ Bonding may be an indirect benefit (not the primary motivation).}\]
2. Adding Storage: Characterization

- Characterization of optimal coalition allocation
  
  - No storage case:
    
    - characterized by consumption floors and decay rates
    - The outside options pin down consumption floors
    - The resource constraint pin down the decay rates
  
  - Storage case:
    
    - characterized by consumption floors, decay rates, and savings
    - (time invariant) $g$ is pinned down by the Euler equation
    - The resource constraint pin down the saving

Proposition

The ergodic ladder is the same as that of the optimal mechanism for any deviating coalition.
Two-State Case - No Storage

- Assumption: $y \in \{y_l, y_h\}$ with $\text{prob}(y_l) = \text{prob}(y_h) = \frac{1}{2}$

- Characterization of ergodic allocation:
  
  - Binding consumption levels: $c_h, c_l$
    
    $$y_h \geq c_h \geq Y \geq c_l \geq y_l, \text{ where } Y = .5(y_h + y_l)$$
  
  - decay rate in the ergodic ladder: $g \geq 1$

- Number of steps in the ladder between $c_h$ and $c_l$
  
    $$T = \arg \max \{ t = 1, \ldots, \infty : c_h/g^{t-1} > c_l \}.$$ 
    
    - If $g > 1$, then $T < \infty$. If perfect insurance is possible $g = 1$, $T = \infty$, and $c_h = c_l = Y$.
    
    - This defines a consumption ladder with $T$ steps

    $$\left\{ c_h, c_h/g, c_h/g^2, \ldots, c_h/g^{T-1}, c_l \right\}.$$
**Two-State Case - No Storage**

- How to determine ergodic $c_h$, $c_l$, and $g$?
  - $c_h$, $c_l$, and $g$ should satisfy the resource constraint and participation constraints.

- Resource constraint w.r.t. $c_h$, $c_l$, and $g$:
  - The ergodic distribution on the $T$ ladder steps:
    \[
    \Pr\left(\frac{c_h}{g^{t-1}}\right) = \left(\frac{1}{2}\right)^t \\
    \Pr(c_l) = 1 - \sum_{t=1}^{T} \left(\frac{1}{2}\right)^t = \left(\frac{1}{2}\right)^T.
    \]
  - Thus, RC should satisfy:
    \[
    \sum_{t=1}^{T} \left(\frac{1}{2}\right)^t c_h/g^{t-1} + \left(\frac{1}{2}\right)^T c_l = Y. \tag{3}
    \]
Two-State Case - No Storage

- Participation constraints w.r.t. $c_h$, $c_l$, and $g$:
  - Determine the payoff conditional on the current consumption level: solving the following system of recursive equations.

  $V_j = u(c_h/g^{j-1}) + \beta \frac{1}{2} [V_{j+1} + V_1]$ for all $j < T + 1$

  $V_{T+1} = u(c_l) + \beta \frac{1}{2} [V_{T+1} + V_1]$.

- PC for the high type:

  $V_1(c_h, c_l, g) = u(y_h) + \beta \left[ \pi \bar{V} + (1 - \pi)V^A \right]$  

- PC for the low type:

  $V_{T+1}(c_h, c_l, g) = u(y_l) + \beta \left[ \pi \bar{V} + (1 - \pi)V^A \right]$  

$\implies$ PC’s depend on endogenous $\bar{V}$ (payoff of coalition).

But how can we determine $\bar{V}$?
Two-State Case - No Storage

- Issues in determining $\overline{V}$:
  - In the transition: time-varying: $c_h(t)$ and $g_t$
  - Why? $g_t \not> g$, as weight of ladder promises press on RC.
- Remark: $c_l$ is always a constant.
  - The conditional payoff at the bottom:
    \[
    V_{T+1} = u(c_l) + \beta \frac{1}{2} [V_{T+1} + V_1],
    \]
  - $V_1$ and $V_{T+1}$ are pinned down by constant outside option.

$\implies$ To determine $\overline{V}$, we need to solve for $\{c_h(t), g_t\}_{t \geq 0}$ and $c_l$.

- **Approximation of $\overline{V}$**: approximate time varying $\{c_h(t), g_t\}_{t \geq 0}$
Two-State Case - No Storage

Approximation

- Assumption: As soon as get $y_h$ start on ergodic ladder $c_h$.
  - For $1 \leq t \leq T - 1$, $\{y_l, y_l, \ldots\}$ type gets residual to satisfy RC:
    
    $$
    c_1 = 2 \left[ Y - \frac{1}{2} c_h \right] 
    $$
    
    $$
    c_t = 2^t \left[ Y - \frac{1}{2} \sum_{j=0}^{t-1} \left( \frac{1}{2g} \right)^j c_h \right], \text{ for } 1 \leq t \leq T - 1
    $$
  
  - In period $T$, everyone is on ergodic ladder ($\{y_l, y_l, \ldots\}$ type hits $c_l$).

$\implies$ Approximated allocation is characterized by ergodic $c_h, c_l$, and $g$.

- Allocation not optimal to the extent that decay rate of $\{y_l, y_l, \ldots\}$ type not $g$.

- Can show decay rate becomes the same if $T$ is large.
**Two-State Case - No Storage Approximation**

- How to determine approximately optimal ergodic $c_h$, $c_l$, and $g$?
  - $c_h$, $c_l$, and $g$ should satisfy two PC’s, and the RC.
  - RC: ergodic RC (3)
    - (Note): By construction, the RC for $t = 1, \cdots, T - 1$ are satisfied.
  - PC:
    - Under the approximation assumption:
      \[
      \bar{V}(c_h, c_l, g) = \sum_{j=1}^{T} \left( \frac{1}{2} \right)^j \beta^{j-1} [V_1 + u(c_j)] + \left( \frac{1}{2} \right)^{T+1} \beta^T [V_1 + V_{T+1}]
      \]
    - $\Rightarrow$ PC’s w.r.t. ergodic $c_h$, $c_l$, and $g$:
      \[
      V_1(c_h, c_l, g) = u(y_h) + \beta \left[ \pi \bar{V}(c_h, c_l, g) + (1 - \pi) V^A \right]
      \]
      \[
      V_{T+1}(c_h, c_l, g) = u(y_l) + \beta \left[ \pi \bar{V}(c_h, c_l, g) + (1 - \pi) V^A \right]
      \]
Two-State Case with Storage

We now consider the case where storage is always used.

[Payoff of initial coalition]

- How to determine ergodic $c_h$, $c_l$, $g$, and $S$?
  - Storage fixes the decay rate to $g = (\beta R)^{-\frac{1}{\gamma}}$.
  - Now outside options + res. const. determine $c_h$, $c_l$ and ergodic $S$.

- How to determine initial $c_1$ and $\{S_t\}$?
  - Ergodic distribution of consumption in $T$ steps so $S_T = S$ ergodic.
  - First period consumption of $y_l$, $c_1$ is determined by $S_T(c_1) = S$.

\[
S_t = 1 - \sum_{j=1}^{t} \frac{c_h}{2^j g^{j-1}} - \frac{1}{2^t} \max \left\{ \frac{c_1}{g^{t-1}}, c_l \right\} + RS_{t-1} \text{ for } t = 1, \cdots T
\]

- $c_h(t)$ and $g(t)$ are constant in the optimal mechanism. Thus, this algorithm is exact. (No approximation)
Two-State Case with Storage

[Payoff of deviating coalition with $y_h$]

- Consumption distribution over time:
  - $t = 1$: $c^d_1 = y_h - s^d_1$, for all HH
  - $t = 2$: $c^d_1/g$, for all HH
    - ...
  - $t = T(c^d_1)$: $c^d_1/g^{T(c^d_1)-1}$, for all HH
  - $t = T(c^d_1) + 1$: $\{c_h, c^d_1/g^{T(c^d_1)}\}$ with prob $\{\frac{1}{2}, \frac{1}{2}\}$
    - ...
  - $t = T(c^d_1) + T(c_h)$: $\{c_h, c_h/g, \cdots, c_h/g^{T(c_h)-1}, \max\{c^d_1/g^{T(c^d_1)+T(c_h)-1}, c_l\}\}$
    - with prob $\{\frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^{T(c_h)}}, \frac{1}{2^{T(c_h)}}\}$
  - $t \geq T(c^d_1) + T(c_h) + 1$: ergodic distribution
  - $S^d_t(c^d_1) = S$ at $t = T(c^d_1) + T(c_h) + 1$ pins down $c^d_1$
  - We can deduce the value of outside option $V_h$. 
Two-State Case with Storage

Payoff of deviation coalition with $y_l$ similar to before

- Lowest possible income so no storage in period 1, $c_1 = y_l$.
- Afterwards follow original optimal mechanism.
- So $V_l = u(y_l) + \beta V = u(y_l) + \beta [\pi \bar{V} + (1 - \pi)V^A]$ optimal coalition payoff.

Constant $g(t)$ and $c_h(t)$ are optimal in the deviating coalition. Thus, this algorithm is exact for the payoff of deviation coalition also.
Two-State Case with Storage

Now two-dimensional fixed point to determine allocation

- Given outside values $V_h$ and $V_l$, solve for $c_h$ and $c_l$ which determines $S$.

- Solve for $\bar{V}$ implied by this ladder, which determines
  \[ V_l = u(y_l) + \beta[\pi \bar{V} + (1 - \pi)V^A] \]
  and iterate until convergence of $V_l$.

- Solve for $V_h$ implied by this ladder and iterate until convergence of $V_h$. 
Two-State Case w. Temporary Storage

Storage may be used but not always. We extend the approximation algorithm.

- Deviating coalition with $y_h$ will go to same ergodic ladder.
  - So if no storage erodically in optimal mechanism no long run storage for them either.
  - But will store temporarily if

$$u'(y_h) < \beta Ru'(c_1).$$

- Decay rate $g_t$ of consumption for $\{y_l, y_l, \ldots\}$ in approximately optimal allocation increase with $t$.
  - Never used in the optimal mechanism if not used ergodically.
  - If used ergodically, it may start to be used in the transition.
For deviating coalition, add simple savings dimension to no storage approximation if not used on ergodic ladder.

- Treat $c$’s as transfers and allocation problem:

$$\begin{align*}
\max_{c_1^d, \{\tau\}} & \quad u(c_1^d) + \sum_{t=2}^{T(c_h)} \beta^{t-1} \left[ \sum_{j=1}^{t-1} u(c_h/g^{j-1} + \tau_{t,j}) \left( \frac{1}{2} \right)^j + u(c_t + \tau_t) \left( \frac{1}{2} \right)^{t-1} \right] \\
& + \sum_{t=T(c_h)+1}^{\infty} \beta^{t-1} \left[ \sum_{j=1}^{T(c_h)} u(c_h/g^{j-1} + \tau_{t,j}) \left( \frac{1}{2} \right)^j + u(c_l + \tau_t) \left( \frac{1}{2} \right)^{T(c_h)} \right] \\
\text{s.t.} & \quad c_1^d + \sum_{t=1}^{T(c_h)} \frac{1}{R^{t-1}} \left\{ \sum_{j=1}^{t-1} \tau_{t,j} \left( \frac{1}{2} \right)^j + \tau_t \left( \frac{1}{2} \right)^{t-1} \right\} \\
& + \sum_{t=T(c_h)+1}^{\infty} \frac{1}{R^{t-1}} \left\{ \sum_{j=1}^{T(c_h)} \tau_{t,j} \left( \frac{1}{2} \right)^j + \tau_t \left( \frac{1}{2} \right)^{T(c_h)} \right\} = y_h, \text{ and } \tau \geq 0.
\end{align*}$$
**Two-State Case w. Temporary Storage**

For the optimal mechanism which uses storage ergodically but not initially, we apply the no storage approximation for initial $k$ periods.

- Ergodic storage fixes the ergodic decay rate to $g = (\beta R)^{-1}$.
- Outside options + res. const. determine ergodic $c_h, c_l$ and $S$.
- For initial $k - 1$ periods where storage is not used, we apply the no storage approximation.
  - As soon as get $y_h$ start on ergodic ladder $c_h$.
  - $\{y_l, y_l, \ldots\}$ type just gets residual from the resource constraint: $\{c_t\}_{t=1}^{k-1}$
- Once storage is used, decay rate is constant $g_t = (\beta R)^{-1}$. Period-$k$ consumption of $y_l, c_k^*$ is determined by RC and need to hit $S$ at $T - 1$.
- Determine $k$ by increasing $k$ until it satisfies $S_{T-1}(c_k) < S$. ($c_k$: residual consumption)
Quantitative Analysis

We take the two-state numerical example with:

- $y_h = 2$, $y_l = 1$
- $\beta = 0.9$
- $\gamma = 1$
- $R = 0$ (no storage) and $R \in [1.01, 1.10]$ (temporary and always)
  (Note: $\beta R < 1$)

We first present the result for $R = 0$ (no storage). Then, we will present the result for storage case.
**Quantitative Analysis ($R = 0$)**

No storage case. Effects of increasing $\pi$:

1. **Payoff (conditional on coalition formation) vs. $\Pi$**
2. **Top and Bottom Consumptions vs. $\Pi$**
3. **Decay rate($g$) vs. $\Pi$**
4. **Step Number($T$) vs. $\Pi$**
Quantitative Analysis ($R = 0$)

- Risk-sharing is declining in $\pi$.
  - Payoff conditional on coalition formation weakly decreasing in $\pi$.
  - Consumption dispersion increases with $\pi$.
  - Decay rate of consumption weakly increasing in $\pi$. Hence growth rate of shadow cost of consumption weakly increasing in $\pi$.
  - Number of steps from in ladder is weakly decreasing in $\pi$.

- Risk sharing can happen even with $\pi = 1$.
  - Comparing $y_h$ followed by $c_h$ or $c_1$ vs. $c_h$ followed by $c_h/g$.

- The approximation error is very small.
**Quantitative Analysis (\(R > 0\)): Payoff**

- Effects of increasing \(\pi\)
  - Payoff conditional on coalition formation is weakly declining in \(\pi\).
  - Ex ante payoff \((\pi\bar{V}(\pi) + (1 - \pi)EV^A)\) has hump shape in \(\pi\) when \(R\) is low. It is increasing in \(\pi\) when \(R\) is high.

\((\pi \uparrow : +, \quad \bar{V}(\pi) \downarrow : -)\)
**Quantitative Analysis ($R > 0$): Payoff**

- **Effects of increasing $R$**
  - For low $R$, storage is only used in outside values, thus increasing $R$ is bad. Once initial coalition uses storage, increasing $R$ is good.
  - For high $\pi$, initial coalition is more likely to use storage at lower $R$. Thus, the turning point $R (- \rightarrow +)$ is decreasing in $\pi$
Quantitative Analysis ($R > 0$): Storage

- At low $R$, storage is only used in deviating coalition and autaky.
- Storage is used in the initial coalition, if $R$ is sufficiently high.
- When $\pi$ is high, shadow price of consumption grows faster over time, and storage can help smoothing shadow prices. Thus, starting point $R(\pi)$ of using storage is decreasing in $\pi$. 

![Storage (Original and Deviating Coalition)](chart.png)
Quantitative Analysis ($R > 0$): Storage

- Saving is weakly increasing in $\pi$ and $R$
Social Efficiency and Savings Efficiency

- Social efficiency and savings efficiency are substitutes when social efficiency ($\pi$) is very low.
- Social efficiency and savings efficiency are complements when social efficiency ($\pi$) is very high.
- In the intermediate level of social efficiency, social and savings efficiency can be either substitutes or complements (substitutes for low $R$, complements for high $R$).
**Quantitative Analysis \((R > 0)\):**

**Allocation**

consumption for the binding guys, decay rate, number of steps:
Basics of Coalitions

Proposition

Infinite coalitions (i.e. \( N = \infty \)) are always optimal.

- Initial coalition everyone ex ante identical - no income in period 0.
  - so can replicate \( N \)-member outcome with \( 2N \) hence weakly better.
- For deviating coalitions
  - Members’ start conditional on \( y_t \). but initial incomes matter only because of impact on per capita \( Y \).
  - Replicate \( N \)-member coalition with same per capita income so weakly better.
- Infinite coalition strictly better since greater insurance.
Basic of Coalitions

**Proposition**

*Breakaway Coalitions will be homogeneous w.r.t. initial income.*

- Prefer coalition members with higher income.
- Since true for everyone, get positive assortative matching in coalition formation - high with highs forces mediums with mediums which in turn forces low with low.
**Two-State Case - No Storage**

Almost optimal if # of steps is large

**Remark**

It then follows that

\[
\frac{c_{t+1}}{c_t} = \frac{2^{t+1} \left[ Y - \sum_{j=1}^{t+1} \left( \frac{1}{2} \right)^j \frac{c_h}{g^j - 1} \right]}{2^t \left[ Y - \sum_{j=1}^{t} \left( \frac{1}{2} \right)^j \frac{c_h}{g^j - 1} \right]} \quad \text{for all } t < T.
\]

Then note that

\[
Y - \sum_{j=1}^{t} \left( \frac{1}{2} \right)^j \frac{c_h}{g^j - 1} = \sum_{j=t+1}^{T} \left( \frac{1}{2} \right)^t \frac{c_h}{g^j - 1} + \left( \frac{1}{2} \right)^T c_l, \quad (4)
\]

where

\[
c_l : c_h / g^{T-1} \geq c_l \geq c_h / g^T.
\]
**Remark**

\[
\frac{c_{t+1}}{c_t} = \frac{1}{g} + \frac{2^{t+1} \left[ (\frac{1}{2})^T c_l - (\frac{1}{2})^{T+1} c_l/g - (\frac{1}{2})^{T+1} c_h/g^T \right]}{2^t \left[ \sum_{j=t+1}^T (\frac{1}{2})^t c_h/g^{j-1} + (\frac{1}{2})^T c_l \right]} > \frac{1}{g}.
\]

However, note that

\[
\frac{2^{t+1} \left[ (\frac{1}{2})^T c_l - (\frac{1}{2})^{T+1} c_l/g - (\frac{1}{2})^{T+1} c_h/g^T \right]}{2^t \left[ \sum_{j=t+1}^T (\frac{1}{2})^t c_h/g^{j-1} + (\frac{1}{2})^T c_l \right]} \simeq 0
\]